Differential Properties of Measures on Infinite Dimensional Spaces and the Malliavin Calculus

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Received 15 March 1989

We study measures induced by smooth functions F on spaces X with smooth (in some sense) measures μ. In particular, we consider the case, when X is a space of functions, μ is a Gaussian measure on X or a measure corresponding to the solution of a stochastic differential equation, and F is a polynomial or a functional of an integral type. We also study infinite dimensional oscillatory integrals of the form $J(t) = \int_X \exp(it\,F(x))\,\mu(\mathrm{d}x)$. Some remarks are made on the relationship between different approaches to Malliavin calculus.

§ 1. Notation and terminology

A Radon measure μ on a locally convex space X (LCS X) is a σ-additive real-valued measure, defined on the Borel σ-algebra $\mathscr{B}(X)$ and satisfying the next condition: for every $A \in \mathscr{B}(X)$ and $\varepsilon > 0$ there exist a compact set $K \subset A$ with $|\mu|(A \setminus K) < \varepsilon$ where $|\mu|$ stands for the total variation of μ. X^* is the topological dual of an LCS X. The Fourier transform of a measure μ is defined by the formula $\tilde{\mu}(l) = \int \exp(il(x))\,\mu(\mathrm{d}x)$, $l \in X^*$. The measure μ is said to be continuous in the direction of a vector $h \in X$ if

$$\lim_{t\to 0} \mu(A + th) = \mu(A)$$

for each $A \in \mathscr{B}(X)$. This is equivalent to the condition $\lim_{t\to 0} \|\mu - \mu_{th}\| = 0$, where $\mu_a(A) = \mu(A + a)$ (see [1]). The measure μ is said to be differentiable in the direction of h if the function $t \mapsto \mu(A + th)$ is differentiable for each $A \in \mathscr{B}(X)$. In this case the set function $A \mapsto \mathrm{d}/\mathrm{d}t\, \mu(A + th)|_{t=0}$ is automatically a bounded measure, called the derivative of μ in the direction of h and denoted by the symbol $\mathrm{d}_h\mu$. The concepts of n-fold and infinite differentiability are defined in the natural way. The partial derivative of μ of the order n in the directions $h_1, \ldots, h_n$ is denoted by the symbol $\mathrm{d}^n_{h_1\ldots h_n}\mu$ (it does not depend upon the order of $h_1, \ldots, h_n$). We say that μ is n-fold

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differentiable in directions of elements of a linear subspace $L \subset X$ if $d^n_{h_1 \dots h_n}\mu$ exists for all $h_1, \dots, h_n \in L$. We say that a measure ν on LCS X has moments of some order p if $q \in L^p(\nu)$ for each continuous seminorm q on X. A mapping $F: X \to Y$ between LCS's X and Y is called differentiable in the direction of h if the limit $\partial_h F(x) = \lim_{t \to 0} (F(x + th) - F(x))/t$ exists.

We use standard notations for spaces of continuous bounded functions, infinitely differentiable functions, infinitely differentiable functions with bounded derivatives and etc.: $C_b(X)$, $C^\infty(R^n)$, $C_b^\infty(R^n)$, $S(R^n)$ stands for Schwarz space of smooth rapidly decreasing functions.

Remark 1. Skorokhod [2] suggested another definition of differentiability. We say that a measure μ is differentiable in the direction of h in the sense of Skorokhod if there exists a measure ν such that for every $f \in C_b(X)$

$$\lim_{t \to 0} t^{-1} \int_X (f(x - th) - f(x)) \, \mu(dx) = \int_X f(x) \, \nu(dx) .$$

As shown in [3], this definition is equivalent to the following: for each $A \in \mathscr{B}(X)$ the function $t \mapsto \mu(A + th)$ is Lipschitzian. Differentiability implies differentiability in the Skorokhod sense and the equality $\nu = d_h\mu$. A measure μ differentiable in the Skorokhod sense is differentiable if and only if the measure ν above is continuous in the direction of h. In turn, it is the case if and only if $\nu \ll \mu$ [4]. The Radon-Nikodym derivative of $d_h\mu$ with respect to μ is denoted by $\varrho_h(\mu)$ and called the logarithmic derivative of μ in the direction of h. Below the Skorokhod derivative is denoted by the same symbol $d_h\mu$.

The discussion of various differential properties of measures on infinite dimensional spaces can be found in [1].

§ 2. Nonlinear images of smooth measures

We shall begin with the following example. Let μ be a measure on R^n, possessing a density $p \in S(R^n)$ with respect to standard Lebesgue measure, and $F: R^n \to R^1$ be a polynomial without critical points (grad $F \neq 0$). It is well-known that the image-measure $\mu \circ F^{-1}$ on R^1, defined by the formula $\mu \circ F^{-1}(B) = \mu(F^{-1}B)$, also has a density $\varrho \in S(R^1)$ and the function $\varphi: t \mapsto \int \exp(it\, F(x)) \, \mu(dx)$ belongs to the same class. The standard way of proving this consists in direct estimations of $\varphi^{(k)}(t) = \int \exp(it\, F(x)) (iF(x))^k \mu(dx) = i^k \int \exp(it\, F(x)) \, F(x)^k \, p(x) \, dx$.

Since $F^k p \in S(R^n)$ it suffices to obtain inequalities of the type $|\psi(t)| \leqq c_m(1 + t^2)^{-m}$, $m \in N$, $\psi(t) = \int \exp(it\, F(x)) \, g(x) \, dx$, $g \in S(R^n)$. For this end we introduce a vector field $v = \text{grad}\, F$ on R^n and consider the operator ∂_v of differentiation along v. Integrating by parts we have:

$$it\, \psi(t) = \int \partial_v(\exp(it\, F(x)) \, (\partial_v F(x))^{-1} \, g(x) \, dx =$$
$$= -\int \exp(it\, F(x)) \, (\partial_v g(x)/\partial_v F(x) - g(x) \, \partial_v^2 F(x)/(\partial_v F(x))^2) \, dx .$$

Notice that $\partial_v g \in S(R^n)$, $g\, \partial_v^2 F \in S(R^n)$, $\partial_v F = (\operatorname{grad} F, \operatorname{grad} F)$ is a polynomial without real zeroes. Hence, by Seidenberg-Tarski theorem (see [5] p. 277) $\partial_v F(x) \geqq c(1 + \|x\|)^{-d}$ for some $c, d > 0$ and $|\partial_v F|^{-p} f \in L^1(R^n)$ for all $p > 0$, $f \in S(R^n)$. Thus, our integration by parts is justified and we can repeat it. Therefore we get the desired estimates.

We are going to study analogous problems for measures μ on infinite dimensional spaces and more general functions F. The first and evident difficulty which we face is nonexistence of any analogue of Lebesgue measure. Certainly, this obstacle could be avoided by defining the action of vector fields directly on measure instead of densities and it will be done below. But the main difficulty which arises here is that the class of vector fields along which a measure is differentiable turns out to be very restricted. As a rule this class does not contain gradients of functionals which are being studied. For example no Gaussian measure γ on an infinite dimensional Hilbert space H can be differentiable along a vector field $v(x) = 2x$ which is the gradient field of the function $F(x) = (x, x)$. Another example: in the infinite dimensional space only zero measure is differentiable (or continuous) in all directions. So, carrying out the programme described above we have to introduce special "smoothed" gradients. The second serious problem is to prove integrability of the function $1/\partial_v F$. The aim of this paper is to discuss the ways of solving these problems. The next abstract definitions will be useful for the sequel.

Definition 2.1. A collection $(X, \mathscr{B}, \mathscr{E})$ is called a measurable manifold if $(X, \mathscr{B})$ is a measurable space and $\mathscr{E}$ is an algebra of bounded $\mathscr{B}$-measurable functions such that for all $f_1, \ldots, f_n \in \mathscr{E}$ and $\varphi \in C^\infty(R^n)$ the composition $\varphi(f_1, \ldots, f_n)$ belongs to $\mathscr{E}$.

Definition 2.2. We say that we have a vector field v on a measurable manifold X if there exists a linear map $\partial_v \colon \mathscr{E} \to L_{\mathscr{B}}$ where $L_{\mathscr{B}}$ is the space of all $\mathscr{B}$-measurable functions, such that

$$\partial_v(\varphi(f_1, \ldots, f_n)) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(f_1, \ldots, f_n)\, \partial_v f_i \tag{2.1}$$

for all $f_1, \ldots, f_n \in \mathscr{E}$, $\varphi \in C^\infty(R^n)$.

Definition 2.3. A measure μ on $\mathscr{B}$ is said to be differentiable along a vector field v if there exists a measure ν on $\mathscr{B}$ such that

$$\int \partial_v f(x)\, \mu(\mathrm{d}x) = -\int f(x)\, \nu(\mathrm{d}x) \tag{2.2}$$

for all $f \in \mathscr{E}$ with $\partial_v f \in L^1(\mu)$.

The measure ν, satisfying (2.2), is called the derivative of μ along v and denoted by $\mathrm{d}_v\mu$. We don't require the uniqueness of $\mathrm{d}_v\mu$, but in the most interesting examples it takes place (it is the case if the class $\mathscr{E}$ distinguishes measures on $\mathscr{B}$). An n-fold differentiability is defined in the natural way. For example we say that a measure μ is twice differentiable along v if $\mathrm{d}_v\mu$ can be taken differentiable. The symbol $\mathrm{d}_v^n\mu$ stands for the derivative of the order n.

We need the following lemmas.

Lemma 2.1. Suppose that an LCS X is a topological sum of an LCS Y and separable Frechet space F, $\{\alpha_i\} \subset F$ and a measure μ on X is n times differentiable along span $\{\alpha_i\}$ (or Skorokhod differentiable, or continuous). Then we can choose conditional measures μ^y on $F + y$ with the same differential properties (correspondently differentiable, Skorokhod differentiable or continuous) such that for all $A \in \mathscr{B}(X)$, $k \leqq n$, $h_1, \ldots, h_k \in \text{span}\, \{a_i\}$

$$d^k_{h_1 \ldots h_k}\mu(A) = \int_Y d^k_{h_1 \ldots h_k}\mu^y(A)\, \nu(\mathrm{d}y) \tag{2.3}$$

where the measure ν on Y is the image of $|\mu|$ under natural projection $X \to Y$.

We omit the proof because it is not difficult and uses the same ideas as in [2], [6] where more special cases were considered.

Corollary. If a measure μ on an LCS X is differentiable in the direction of $h \in X$, $f \in L^1(d_h\mu)$ and $\partial_h f \in L^1(\mu)$ then the following equality holds:

$$\int \partial_h f(x)\, \mu(\mathrm{d}x) = -\int f(x)\, d_h\mu(\mathrm{d}x) \tag{2.4}$$

Proof. There exists an LCS Y such that $X = Y + R^1 h$. Hence by lemma 2.1 we can consider the case $X = R^1$. In this case μ has an absolutely continuous density p with $p' \in L^1(R)$. Besides we have inclusions $f'p \in L^1(R)$, $p'f \in L^1(R)$ (since for almost all $y \in Y$ the function $t \mapsto \partial_h f(y + th)$ is μ^y-integrable and the function $t \mapsto f(y + th)$ is $d_h\mu^y$-integrable). In our special case the formula (2.4) is almost evident, but the accurate proof is not, however, very short, because we have to verify absolute continuity of fp in the case when f need not be absolutely continuous and p need not be differentiable everywhere. So we don't enter the details, but only mention that the desired statement may be deduced from the fact that everywhere differentiable on $[a, b]$ function g is absolutely continuous if $g' \in L^1[a, b]$ (see [7]). This fact is applied to segments on which p has no zeros.

Lemma 2.2. If μ is a measure on R^1 ,

$$r \in N\,, \quad p \geqq 1\,, \quad \frac{1}{p} + \frac{1}{q} = 1$$

and for all $\varphi \in C_0^\infty(R^1)$, $k \leqq r$

$$\left|\int \varphi^{(k)}(t)\, \mu(\mathrm{d}t)\right| \leqq M(r, p)\, \|\varphi\|_{L^p} \tag{2.5}$$

then μ admits a density f, which is $r - 1$ times differentiable, $f^{(k)} \in L^q(R^1)$ $(k \leqq r - 1)$; $f^{(r-1)}$ has the bounded variation if $p = \infty$.

Corollary. If $p = \infty$ and $J(t) = \int \exp(itx)\, \mu(\mathrm{d}x)$ then

$$|J(t)| \leqq \operatorname{Var} f^{(r-1)} |t|^{-r} = \|d_1^r \mu\|\, |t|^{-r}\,.$$

The proof is standard and can be found, for example, in [8].

Remark 2.1. If μ is a measure on R^n and (2.5) holds for all $\varphi \in C_0^\infty(R^n)$ where $\varphi^{(k)}$ means $\partial_{i_1} \dots \partial_{i_k}\varphi$, $\partial_j = \partial/\partial x_j$, $i_1, \dots, i_k \in \{1, \dots, n\}$, then the conclusion of the lemma is also true, but $f^{(k)}$ should be understood in the Sobolev sense (hence if the condition above holds for each r then f can be taken in $C_b^\infty(R^n)$).

For the sequel it is convenient to extend the notion of differentiability of a function on an abstract measurable manifold X with a measure μ. But first we consider examples of such manifolds.

Example 1. Let X be an LCS with a Radon measure μ, $\mathscr{B} = \mathscr{B}(X)$, $h \in X$, $\mathscr{E}$ consists of all functions of the form $f(x) = \varphi(\pi(x))$ where $\pi: X \to R^n$ is a continuous linear mapping and $\varphi \in C_b^\infty(R^n)$ (or $\mathscr{E}$ can consist of all bounded Borel functions which are infinite differentiable in the direction of h). Then μ is differentiable along the vector field h, defined by the operator $f \mapsto \partial_h f$, if and only if μ is Skorokhod differentiable in the direction of h.

Example 2. Suppose that M is a finite dimensional smooth manifold with the Borel σ-algebra $\mathscr{B}$, $\mathscr{E}$ is the algebra of bounded C^∞-functions on M, a measure λ on M has a compact support and in local coordinates is defined by a smooth density. Then λ is differentiable along each continuous vector field on M.

Example 3. If $(X_n, \mathscr{B}_n, \mathscr{E}_n)$ are measurable manifolds with probability measures m_n, differentiable along vector fields v_n, then the space

$$\left(X = \bigotimes_{n=1}^{\infty} X_n\,,\ \bigotimes_{n=1}^{\infty} \mathscr{B}_n\right)$$

has a structure of measurable manifold if for $\mathscr{E}$ we take a class of all functions of the form $f = \varphi(f_1 \circ \pi_1, \dots, f_n \circ \pi_n)$ where $\varphi \in C^\infty(R^n)$, $f_i \in \mathscr{E}_i$, $\pi_i: X \to X_i$ are natural projections. The measure $m = \bigotimes_{n=1}^{\infty} m_n$ is differentiable along any vector field

$$\partial_v = \sum_{i=1}^{n} \partial v_i$$

defined by the formula

$$\partial_v f = \sum \frac{\partial \varphi}{\partial x_i}(f_1 \circ \pi_1, \dots, f_n \circ \pi_n)\, \partial v_i f_i \circ \pi_i\,.$$

Under some additional assumptions it is possible to define a vector field

$$\sum_{i=1}^{n} \partial_{v_i}\,.$$

If $X_n \equiv S^1$ then X can be called an infinite dimensional torus. Notice that in the last case the measurable manifold X has no structure of Banach manifold or LCS-modeled manifold.

Now we return to the problem of extension of the class of differentiable functions. The standard way of such extension is to consider limits (for example in L^p) of sequences of functions with derivatives converging with respect to some norm (say in L^r). For this see [8], [9]. In this paper we choose the simplest way (for some problems it is not the best): we say that f is an infinitely differentiable function on X if there exists a sequence of functions $f_n \in \mathscr{E}$ converging to f in every $L^p(\mu)$ such that $\partial_v^k f_n \in \mathscr{E}$ for each k and $\{\partial_v^k f_n\}$ are fundamental in all $L^p(\mu)$.

Theorem 2.1. Suppose that X is a measurable manifold, a measure μ is infinitely differentiable along vector fields $v_1, \ldots, v_n$, measurable functions $\varphi_1, \ldots, \varphi_n$ are infinitely differentiable along $v_1, \ldots, v_n$ (in the sense mentioned above), $F = (\varphi_1, \ldots, \varphi_n)$, $d^r_{i_1 \ldots i_r}\mu = \varrho(i_1, \ldots, i_r)\,\mu$, $\varrho(i_1, \ldots, i_r) \in \bigcap_p L^p(\mu)$, $\sigma_{ij} = \partial_{v_i}\varphi_j$. If $\Delta = \det(\sigma_{ij}) \neq 0$ almost everywhere and

$$\Delta^{-1} \in \bigcap_p L^p(\mu)$$

then the measure $\mu \circ F^{-1}$ on R^n has a density $p \in C_b^\infty(R^n)$.

Proof. Denote by (γ^{ij}) the inverse matrix for (σ_{ij}) and by A^{ij} the signed minor of the element σ_{ij} in the matrix (σ_{ij}). Then $\gamma^{ik} = A^{ik}/\Delta$ and $\partial_i \gamma^{jk} = (\Delta\, \partial_i A^{jk} - A^{jk}\, \partial_i A)/\Delta^2$, where we write ∂_i instead of ∂_{v_i}. By the condition of the theorem

$$\partial_i \gamma^{jk} \in \bigcap_p L^p(\mu)\,.$$

Let f be in $C_0^\infty(R^n)$. Then μ-a.e.

$$\frac{\partial f}{\partial x_i} \circ F(y) = \sum_{k,m} \gamma^{ik}(y)\, \sigma_{km}(y) \frac{\partial f}{\partial x_m} \circ F(y) = \tag{2.6}$$

$$= \sum_{k,m} \gamma^{ik}(y)\, \partial_k \varphi_m(y) \frac{\partial f}{\partial x_m} \circ F(y) = \sum_k \gamma^{ik}(y)\, \partial_k (f \circ F)(y)\,.$$

If φ and ψ are infinitely differentiable along v_k then it is easy to see that $\partial_k(\varphi\psi) = \varphi\, \partial_k \psi + \psi\, \partial_k \varphi$ and

$$\int \partial_k(\varphi\psi)(x)\, \mu(\mathrm{d}x) = -\int \psi(x)\, \varphi(x)\, d_k\, \mu(\mathrm{d}x)\,.$$

For the proof it is sufficient to establish these equalities for $\varphi, \psi \in \mathscr{E}$. In this case the second equality follows from the first one and the definition, and the first one is easily deduced from the chain rule indicated in Definition 2.2. Therefore we get the following integration by parts formula:

$$\int \varphi(x)\, \partial_k \psi(x)\, \mu(\mathrm{d}x) = -\int \psi(x)\, \partial_k \varphi(x)\, \mu(\mathrm{d}x) - \int \varphi(x)\, \psi(x)\, d_k \mu(\mathrm{d}x)\,. \tag{2.7}$$

Let Φ be infinitely differentiable along v_k. According to (2.6) and (2.7) we have:

$$\int \frac{\partial f}{\partial x_i} \circ F(y)\, \Phi(y)\, \mu(\mathrm{d}y) = \sum_k \int \gamma^{ik}(y)\, \partial_k(f \circ F)\, \Phi(y)\, \mu(\mathrm{d}y) = \tag{2.8}$$

$$= - \sum_k \int f \circ F(y)\, \partial_k(\gamma^{ik}\Phi)\,(y)\,\mu(\mathrm{d}y) - \sum_k \int f \circ F(y)\,\gamma^{ik}(x)\,\Phi(y)\,d_k\,\mu(\mathrm{d}y) =$$

$$= - \sum_k \int f \circ F(y)\,\partial_k\gamma^{ik}(y)\,\Phi(y)\,\mu(\mathrm{d}y) - \sum_k \int f \circ F(y)\,\gamma^{ik}(y)\,\partial_k\Phi(y)\,\mu(\mathrm{d}y) -$$

$$- \sum_k \int f \circ F(y)\,\gamma^{ik}(y)\,\Phi(y)\,d_k\,\mu(\mathrm{d}y)\,.$$

Let r be a fixed natural number. Take $\Phi \equiv 1$ and apply (2.8) to $\partial^r f/(\partial x_{i_1} \dots \partial x_{i_r})$ instead of $\partial f/\partial x_i$. In the right hand of (2.8) we obtain the sum of expressions of the form $\int g \circ F(y)\, G(y)\, \lambda(\mathrm{d}y)$, where $g = \partial^{r-1} f/\partial x_{i_1} \dots \partial x_{i_{r-1}}$, G is a polynomial of γ^{ik} and $\partial_k \gamma^{ik}$, $\lambda = d_i\mu$. The conditions of the theorem permit to apply (2.8) to these expressions. Having fulfilled this procedure r times we represent the integral

$$\int \frac{\partial^r f}{\partial x_{i_1} \dots \partial x_{i_r}} \circ F(y)\,\mu(\mathrm{d}y)$$

as the sum of integrals of the type $\int f \circ F(y)\, G(y)\, \lambda(\mathrm{d}y)$ where G is a polynomial of γ^{ik} and mixed derivatives of γ^{ik} along v_k (of the order less than $r + 1$), λ is a mixed derivative of μ along v_k (also of the order less than $r + 1$). Now the desired statement follows from Lemma 2.2 and Remark 2.1.

Remark 2.2. It is clear from the proof that sufficient conditions of r-fold differentiability of the density p can be obtained using restrictions on derivatives of φ_i and μ of orders $s \leqq r + 1$.

Denote by H_α the space of all functions on $[a, b]$ satisfying the Holder condition of the order $\alpha \in (0, 1]$, and by W_m the space of all functions f on $[a, b]$ with absolute continuous $f^{(m-1)}$ and $f^{(m)} \in L^2[a, b]$, $f(0) = 0$. Recall that a Radon measure γ on an LCS X is said to be Gaussian if each continuous linear functional l has Gaussian distribution on (X, γ). If all $l \in X^*$ have mean zero then γ is called symmetric. It is known that the Fourier transform of a symmetric Gaussian measure γ on a Hilbert space H can be written as $\tilde{\gamma}(y) = \exp(-(Ay, Ay))$, where A is nonnegative Hilbert-Schmidt operator. The subspace $H(\gamma) = A(H)$ is called the reproducing kernel of γ. It coincides with the subspace $D(\gamma)$ of all vectors in directions of which γ is differentiable. We are going to present applications of our general Theorem 2.1 to smooth functions defined on locally convex spaces with Gaussian or more general differentiable measures. We shall consider vector fields of the form

$$v(x) = \sum_{n=1}^{\infty} \alpha_n\, \varphi_n(x)\, a_n$$

where $\alpha_n > 0$, φ_n are „smooth" (in some sense) functions and vectors a_n are such that our measure is differentiable in their directions. Notice that according to the corollary of Lemma 2.2 the measure μ is differentiable along a vector field $v(x) = \varphi(x)\, h$ provided that φ and μ are differentiable in h and $\partial_h \varphi \in L^1(\mu)$, $\varphi \in L^1(d_h\mu)$. In view of what has been said it is clear that the structure of the subspace $D(\mu)$ of all vectors along which μ is differentiable has great importance for our considerations. The next result, obtained in [10], clarifies the situation.

Theorem 2.2. Let μ be a Radon measure on an LCS X, $D(\mu)$ and $D_C(\mu)$ are subspaces in X consisting of all vectors along which μ is differentiable and Skorokhod differentiable. Then $D_C(\mu)$ can be equipped with the norm p possessing the following properties:

1) $(D_C(\mu), p)$ is a Banach space isomorphic to a conjugate space Y^*,
2) natural embedding of $D_C(\mu)$ in X is a compact operator (that is: a closed unit ball of $D_C(\mu)$ is compact in X),
3) $D(\mu)$ is a closed subspace in $D_C(\mu)$,
4) for p we can take $p(h) = \|d_h\mu\|$.

The fact that $D_C(\mu)$ is isomorphic to a conjugate space is not indicated in [10] but follows immediately from Mackey's theorem and compactness of the set $\{h \in D_C(\mu): \|d_h\mu\| \leqq 1\}$ in X which results from Theorem 5 in [10] or Theorem 1 in [3].

Theorem 2.3. Let γ be a symmetric Gaussian measure on $C_0[a, b]$ such that for some α, m_0 $\gamma(H_\alpha) = 1$ and $W_{m_0} \in D(\gamma)$, $F(x) = \int_a^b f(x(t))\, dt$ where $f \in C^{r+2}(R)$ satisfies the following conditions:

1) $|f^{(r+2)}(t)| \leqq c \exp(d|t|)$,
2) for some $\beta, \nu, \delta > 0$ $|f'(t)| \geqq \beta|t|^\nu$ if $|t| \leqq \delta$. Then $\gamma \circ F^{-1}$ has a density p with integrable derivatives $p', \ldots, p^{(r)}$ and $p^{(r)} \in BV(R)$.

Proof. We shall regard γ on $H = L^2[a, b]$. Without any loss of generality we can assume that $[a, b] = [0, 1]$. The closed graph theorem together with Theorem 2.2 and inclusion $W_{m_0} \subset D(\gamma)$ gives us the following inequality which holds for all $h \in W_{m_0}$:

$$\|d_h\gamma\| \leqq c\|h\|_{m_0}. \tag{2.9}$$

Here by $\|\cdot\|_m$ we mean $\|h\|_m = \|h\|_{L^2} + \|h^{(m)}\|_{L^2}$. It is easy to show that γ is infinitely differentiable along $D(\gamma)$ and for every $r \in N$ there exists $d(r)$ such that for all $h_1, \ldots, h_r \in D(\gamma)$

$$\|d^r_{h_1\ldots h_r}\gamma\| \leqq d(r)\, \|d_{h_1}\gamma\| \cdot \ldots \cdot \|d_{h_r}\gamma\|, \tag{2.10}$$

(it is possible to take $d(r) = r^{r/2}$, see [1] for details). Fix $r \in N$ and $m > 2m_0 r$. Let $e_n(t) = \sqrt{2}\sin((n - \frac{1}{2})\pi t)$ (see [11]) ($\{e_n\}$ is a base in $L^2[0, 1]$, in particular $\|e_n\|_m \leqq n^m + 1$) and define an operator A in H by $Ae_n = \alpha_n e_n$, $\alpha_n = n^{-m}$. The vector field

$$v(x) = \sum_{n=1}^{\infty} \alpha_n^2\, \partial_{e_n}F(x)\, e_n$$

will play the main role in our proof. Notice that $\partial_h F(x) = \int_a^b f'(x(t))\, h(t)\, dt$. Hence F

is $r+2$ times differentiable along v and for $k \leqq r+2$ $\partial_v^k F \in \bigcap_p L^p(\gamma)$. From the considerations above it is clear that γ is $r+2$ times differentiable along v and $d_v^2\gamma = \varrho(k, v)\,\gamma$ with $\varrho(k, v) \in \bigcap_p L^p(\gamma)$. Indeed $d_v\gamma = \sum \alpha_n^2\, \partial_{e_n}^2 F\gamma + \sum \alpha_n^2\, \partial_{e_n} F d_{e_n}\gamma$. According to (2.9), (2.10) for $h_1, \dots, h_k \in \{e_1, \dots, e_s\}$

$$\|d_{h_1\dots h_k}^k \gamma\| \leqq c(k.\, m_0)\, s^{m_0}\,.$$

This estimate permits to differentiate $d_v\gamma$ along v because after differentiation we obtain the series converging in variation:

$$d_v^2\gamma = \sum_k \alpha_k^2\, \partial_k^2 F d_v\gamma + \sum_k \alpha_k^2\, \partial F\Big(\sum_n \alpha_n^2\, \partial_k\, \partial_n F d_n\gamma +$$

$$+ \alpha_n^2\, \partial_n F d_k d_n\gamma + \alpha_n^2\, \partial_k \partial_n^2 F\gamma + \alpha_n^2\, \partial_n^2 F d_k\gamma\Big)\,, \quad \partial_i = \partial_{e_i}\,.$$

Now we should examine the function $\partial_v F(x) = \sum_n \alpha_n^2 (\partial_{e_n} F(x))^2$ and prove estimates $\gamma(x\colon \partial_v F(x) < \varepsilon) \leqq \varkappa_k \varepsilon^k$, $k \in N$. Possessing these estimates we can apply the method of Theorem 2.1 and get r-fold differentiability of the density of $\gamma \circ F^{-1}$. For $x \in C_0[a, b]$ $\partial_v F$ can be represented in the following form: $\partial_v F(x) = (\Lambda \operatorname{grad} F(x), \Lambda \operatorname{grad} F(x))$ where $(\operatorname{grad} F(x), h) = \int_a^b f'(x(t))\, h(t)\, dt$. Therefore, denoting by S the unit ball in H we have:

$$\partial_v F(x) = \|\Lambda \operatorname{grad} F(x)\|^2 = \sup_{h\in S} (\Lambda \operatorname{grad} F(x), h)^2 =$$

$$= \sup_{h\in S} (\operatorname{grad} F(x), \Lambda h)^2 = \sup_{y \in \Lambda(S)} (\operatorname{grad} F(x), y)^2\,.$$

The set $\Lambda(S)$ contains the unit ball S_m in W_m (with the norm $\|\cdot\|_m$). It is seen from direct calculations. Hence

$$\partial_v F(x) \geqq \sup_{h \in S_m} \Big(\int f'(x(t))\, h(t)\, dt\Big)^2\,.$$

Denote by p_α the standard norm in H_α:

$$p_\alpha(x) = |x(0)| + \sup_{t,s} |x(t) - x(s)|/|t - s|^\alpha\,.$$

Introduce sets $S(\varepsilon) = \{x \in C_0[a, b]\colon \|x\|_H \leqq \varepsilon\}$ and $E(\varepsilon) = \{x \in H_\alpha\colon p_\alpha(x) \leqq 1/\varepsilon\}$. Since $\gamma(H_\alpha) = 1$ then by Fernique's theorem (see [11]) $p_\alpha \in \bigcap_p L^p(\gamma)$ and hence for every $k \in N$ we can choose η_k such that for all ε

$$\gamma(E(\varepsilon)) \geqq 1 - \eta_k \varepsilon^k\,.$$

The condition $W_m \subset D(\gamma)$ gives smoothness of finite-dimensional projections of γ. This implies the following estimates:

$$\gamma(S(\varepsilon)) \leqq \xi_k \varepsilon^k\,.$$

Let $x \in M(\varepsilon) = E(\varepsilon) \setminus S(\varepsilon)$ and $\varepsilon < \min(\delta, \frac{1}{4})$. Then there exists $t_0 \in [a, b]$ such that $|x(t_0)| = \varepsilon/2$ and for $|t| \leqq \varepsilon^{3/\alpha}$ $|x(t_0 + t) - x(t_0)| \leqq \varepsilon^{-1}|t|^\alpha \leqq \varepsilon^2$. Thus, $|x(t)| \in \in [\varepsilon/4, \varepsilon]$ for $t \in [t_0 - \varepsilon^{3/\alpha}, t_0 + \varepsilon^{3/\alpha}]$ and so $f'(x(t)) \geqq \beta(\varepsilon/4)^\nu$ or $f'(x(t)) \leqq -\beta(\varepsilon/4)^\nu$. Fix a function $\varphi \in C^\infty$ with the following properties: $0 \leqq \varphi \leqq 1$, $\operatorname{supp} \varphi \subset [-1, 1]$, $\varphi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Let $C_m = (2 + 2 \sup |\varphi^{(m)}|)^{-1}$. For the above mentioned x and t_0 define h by the formula:

$$h(t) = c_m \varepsilon^{3m/\alpha} \varphi((t - t_0)/\varepsilon^{3/\alpha}).$$

Then $\|h\|_m \leqq 1$ and we have the following inequality:

$$\left(\int_a^b f'(x(t))\, h(t)\, \mathrm{d}t\right)^2 \geqq (\beta(\varepsilon/4)^\nu c_m \varepsilon^{3m/\alpha} \varepsilon^{3/\alpha})^2 = (\beta c_m 4^{-\nu})^2 \varepsilon^d$$

where $d = (6m + 6)/\alpha + 2\nu$. Therefore we have proved the estimate

$$\gamma(x\colon \partial_v F(x) < (\beta c_m 4^{-\nu})^2 \varepsilon^d) \leqq (\eta_k + \xi_k)\, \varepsilon^k$$

with β, c_m, ν, d not depending on k and ε. This completes the proof.

Remark 2.3. The analogous theorem can be proved for measures on the space of vector-valued functions (that is, for distributions of vector-valued processes). It is clear from the proof that the same is valid for more general (not necessarily Gaussian) measures on $C_0[a, b]$, differentiable along W_m, provided some additional restrictions concerning the moments are imposed.

Corollary. $\int \exp(it\, F(x))\, \gamma(\mathrm{d}x) = 0(|t|^{-r-1})$ when $t \to \infty$.

Remark 2.4. The last corolary can be used to prove the following result: if functions $f_1, \ldots, f_n$ are such that for all $(\alpha_1, \ldots, \alpha_d) \neq (0, \ldots, 0)$ the function $\alpha_1 f_1 + \ldots + \alpha_n f_n$ satisfies the conditions of Theorem 2.3 (for each r) then the mapping $F = (F_1, \ldots, F_n)$: $X \to R^n$ induces the measure $\gamma \circ F^{-1}$ on R^n with the density $p \in S(R^n)$ (so $F_1, \ldots, F_n$ have smooth joint distribution).

Our previous results have the following character: finite-dimensional smooth images of differentiable measures are also differentiable. The natural question arises whether the same is true for infinite dimensional images. In general the answer is negative: even a polynomial of the second order on a Hilbert space can transform a Gaussian measure in a nondifferentiable measure (see [1] for details). Finite-dimensional polynomial mappings are discussed in § 3. If we are interested only in the existence of densities or their boundedness the conditions of the theorems above can be considerably weakened. The results of this type are discussed in § 4. The next lemma is also a statement of this sort.

Lemma 2.3. Let μ be a measure on R^n continuous in the directions of the standard base $e_1, \ldots, e_n$. Then μ admits a density p with respect to Lebesgue measure. If, moreover, the measure $d^n_{e_1 \ldots e_n}\mu$ exists (even if in the Skorokhod sense) then p can be chosen bounded with $\sup |p(x)| \leqq \|d^n_{e_1 \ldots e_n}\mu\|$.

Proof. The first assertion is trivial (see [1]). The second one can be proved by approximating μ in variation by convolutions $\mu * \gamma$ with Gaussiah measure γ (see [12] for details) and verifying the desired estimate for measures.

§ 3. Distributions of polynomials and infinite dimensional oscillatory integrals

A polynomial on an LCS X is a function of the form

$$P(x) = \sum_{k=0}^{n} V_k(x, \ldots, x)$$

where $V_k\colon X^k \to R^1$ are continuous k-linear forms (we are discussing real-valued polynomials; polynomial mappings between locally convex spaces are defined in the same way, see [13] for equivalence of different natural definitions). The function $Q(x) = V_2(x, x)$ is called a quadratic form on X. It can be written in the following way: $Q(x) = (Ax)(x)$ where $A\colon X \to X^*$ is a symmetric linear map. If $\dim A(X) = \infty$ the form Q is said to be infinite dimensional.

Lemma 3.1. Given $d \in N$ there exists $c(d) > 0$ such that for every polynomial $P(t) = t^d + \ldots$ and each measure μ on R^1

$$|\mu|\,(t\colon |P(t)| \leqq \varepsilon) \leqq c(d)\, \|d_1\mu\|\, \varepsilon^{1/d}\,. \tag{3.1}$$

Proof. This follows from Lemma 2.3 and the estimate (see [14]):

$$\operatorname{mes}(t\colon |P(t)| \leqq \varepsilon) \leqq c(d)\, \varepsilon^{1/d}\,.$$

Theorem 3.1. Suppose that a Radon measure μ and a polynomial Q of the degree d on an LCS X satisfy the following conditions:

1) X is the topological sum of LCS's $X_0, \ldots, X_n$ and $Q(x) = Q_0(x) + \ldots + Q_n(x)$ where Q_i are polynomials depending only on the projections of x on X_0 and X_i;
2) there exist such $h_i \in X_i$, $i = 1, \ldots, n$, that $\deg Q_i = \deg Q_i|_{R^1 h_i} > 0$ and μ is $r + n$ times differentiable along $\operatorname{span}(h_1, \ldots, h_n)$ and all its derivatives of orders $s \leqq r$ have moments of the order $2r^2(d-1)$;
3) $n > 4(d-1)(r+1)$. Then $\mu \circ Q^{-1}$ admits a density p with integrable $p', \ldots, p^{(r-1)}$ and $p^{(r-1)} \in BV(R^1)$.

Proof. We give only a sketch of the proof. Define a vector field v on X by the formula

$$v(x) = \sum_{i=1}^{n} \partial_{h_i} Q(x)\, h_i\,.$$

Q is infinitely differentiable along v. It is not difficult to check r-fold differentiability of μ along v. Acting in the same way as in the proof of Theorem 2.1 we meet our familiar problem of integrability of $1/\partial_v Q$. In the present case

$$G(x) = \partial_v Q(x) = \sum_{i=1}^{n} (\partial_{h_i} Q(x))^2$$

is a polynomial of the degree $\varrho \leqq 2d - 2$ with the following properties: $G(x) = G_1(x) + \ldots + G_n(x)$, where non-zero polynomials G_i depend only on the projections of x on X_0 and X_i, $\deg G_i = \deg G_i|_{R^1 h_i}$. For our aims it suffices to get the estimate

$$|v| (x: G(x) \leqq \varepsilon) \leqq c(v, G)\, \varepsilon^{n/\varrho} .$$

for every measure v n times differentiable along span $(h_1, \ldots, h_n)$. Use the induction. Let $E = \{x \in X_0 + \ldots + X_{n-1}\colon G_1(x) + \ldots + G_{n-1}(x) \leqq \varepsilon\}$, λ be the natural projection of $|v|$ on $Y = X_0 + \ldots + X_{n-1}$. According to Lemma 2.1 it is possible to choose conditional measures v^y on $X_n + y$ to be differentiable in h_n. Hence by Lemma 3.1

$$|v| (x: G(x) \leqq \varepsilon) \leqq \int_E |v^y| (x: G_n(x) \leqq \varepsilon)\, \lambda(\mathrm{d}y) \leqq$$
$$\leqq c(\deg G_n)\, \varepsilon^{1/\varrho} \int_E \|d_{h_n} v^y\|\, \lambda(\mathrm{d}y) =$$
$$= c(\deg G_n)\, \varepsilon^{1/\varrho} |d_{h_n} v| (x: G_1(x) + \ldots + G_{n-1}(x) \leqq \varepsilon) \leqq \operatorname{const}(G, v)\, \varepsilon^{n/\varrho} ,$$

because $d_{h_n} v$ is $n - 1$ times differentiable along $h_1, \ldots, h_{n-1}$ and the inductive assumption is applicable. It remains to notice that $d_v^r \mu$ is n times differentiable along $h_1, \ldots, h_n$. Thus, we have integrability of inverse powers of $\partial_v Q$ and can realize the method of Theorem 2.1.

Corollary 1. If the conditions of Theorem 3.1 are fulfilled for all n and r then $p \in S(R^1)$ and for each k

$$\int \exp(it\, Q(x))\, \mu(\mathrm{d}x) = 0(|t|^{-k}) , \quad t \to \infty .$$

Corollary 2. Let $F_1, \ldots, F_d$ be polynomials such that for all $(\alpha_1, \ldots, \alpha_d) \neq (0, \ldots, 0)$ polynomials $\alpha_1 F_1 + \ldots + \alpha_d F_d$ satisfy conditions of Theorem 3.1 (for all n, r). Then the mapping $F = (F_1, \ldots, F_d)$ induce a measure on R^d with a density belonging to $S(R^d)$.

Corollary 3. Let $X = L^\infty(T)$, where T is a space with a finite measure σ, supp $\sigma = T$, $Q(x) = \int q(t, x(t))\, \sigma(\mathrm{d}t)$, $q(t, z) = z^n + a_{n-1}(t)\, z^{n-1} + \ldots + a_0(t)$, $a_i \in L^1(T)$. Suppose that γ is a Gaussian measure on X and for each m there exist $h_1, \ldots, h_m \in D(\gamma)$ with disjoint supports. Then the conclusion of Corollary 1 is true.

For the proof of the last corollary take $X_i = R^1 h_i$ and denote by X_0 a topological complement to span $(h_1, \ldots, h_m)$ (it exists).

Example 3.1. Suppose that μ is a measure on R^n with a density $g \in S(R^n)$, $F = (F_1, \ldots, F_d)\colon R^n \to R^d$, F_i are polynomials and g vanishes with all derivatives on the set $Z = \{x\colon \Delta(x) = 0\}$ where $\Delta(x) \det((\operatorname{grad} F_i, \operatorname{grad} F_j)_{i,j=1}^d)$. Then the measure $\mu \circ F^{-1}$ on R^d has a density $p \in S(R^d)$.

Proof. Consider vector fields $v_i = \operatorname{grad} F_i$. Let $\sigma_{ij} = \partial_{v_i} F_j = (\operatorname{grad} F_i, \operatorname{grad} F_j)$. Applying ∂_{v_i} to a measure λ with a density $\psi \in S(R^n)$ vanishing with all derivatives on Z we obtain the measure with the same properties. So the only thing we need to verify is the inclusion $\Delta^{-1} \in \bigcap_p L^p(\lambda)$ where the measure λ has a density $\psi \in S(R^n)$ vanishing with derivatives on Z. For each $m \in N$ there exists $c_m > 0$ such that $|\psi(x)| \leqq c_m \operatorname{dist}(x, Z)^m$. By Seidenberg-Tarski theorem

$$\Delta(x) \geqq c(\operatorname{dist}(x, Z))^\alpha (1 + \|x\|)^{-\beta}, \quad \alpha, \beta \geqq 0. \tag{3.2}$$

The desired statement directly follows from these estimates and Theorem 2.1.

The example considered in § 1 is a special case of Example 3.1. It is unknown whether the analogous statement is valid for infinite dimensions (even for polynomials without critical points). In this connection it is important to stress that in Seidenberg-Tarski's theorem one can't obtain the estimate (3.2) with β not depending on the dimension of the space R^n (see [15]). In view of what has been said above it is clear that the realization of our programme depends considerably on successful solving of the problem of finding asymptotics of expressions $\mu(x\colon |G(x)| \leqq \varepsilon)$ where μ is a smooth measure on an infinite dimensional space and G is a regular function. Our investigations are also closely connected with studying of oscillatory integrals $t \mapsto \int \exp(it F(x)) \mu(dx)$. In finite-dimensional situation similar questions are much more worked out but even in this case not everything is clear (some positive results and further discussion can be found in [15]). On the one hand the infinite dimensional case is more complicated but on the other hand in this case Gaussian (or smooth) measures of small balls decrease very quickly so that we get some „compensation". In the next two theorems we shall see how this circumstance can work.

Theorem 3.2. Suppose that a Radon measure μ on an LCS X is infinitely differentiable along a dense linear subspace $D \subset X$ and all its partial derivatives have moments of all orders. If quadratic forms $Q_1, \ldots, Q_n$ on X are such that for each $(\alpha_1, \ldots, \alpha_n) \neq 0$ the form $\alpha_1 Q_1 + \ldots + \alpha_n Q_n$ is infinite dimensional then the map $Q = (Q_1, \ldots, Q_n)$ induces the measure, possessing a density $p \in S(R^n)$. In particular it is true for each nondegenerate Gaussian measure μ on X.

Proof. Again we only sketch the proof. It suffices to show that the function

$$\varphi(t_1, \ldots, t_n) = \int \exp\Big(i \sum_{j=1}^{n} t_j\, Q_j(x)\Big) \mu(dx)$$

belongs to $S(R^n)$. Partial derivatives of φ have the form $i^k \int \exp(i \sum t_j\, Q_j(x)) \nu(dx)$ where $\nu = Q_{j_1} \cdot \ldots \cdot Q_{j_k} \mu$ possesses the same properties as μ. Thus, we need only estimates

$$|\varphi(t_1, \ldots, t_n)| \leqq c_k(\|t\| + 1)^{-k}.$$

For this end consider first the case $n = 1$ and take vectors $a_1, \ldots, a_r \in D$ with linearly independent $l_1 = Aa_1, \ldots, l_r = Aa_r$. Define a vector field

$$v(x) = \sum_{i=1}^{r} \partial_{a_i} Q(x)\, a_i = 2 \sum_{i=1}^{r} l_i(x)\, a_i\,.$$

Then

$$\partial_v Q(x) = 2 \sum_{i=1}^{r} (l_i(x))^2$$

is a quadratic form non degenerating on $L = \operatorname{span}(a_1, \ldots, a_r)$. It is clear that μ is infinitely differentiable along v and by induction we obtain that $d_v^k \mu$ is a finite sum of measures of the form $v = \Phi_j(l_1(x), \ldots, l_r(x))\, d^j_{i_1 \ldots i_j} \mu$ where Φ_j is a polynomial, $\deg \Phi_j \leqq j$. For example

$$d_v^2 \mu = 4(\sum_i l_i(a_i))^2 \mu + (4 \sum_i l_i(a_i)) \sum_i l_i d_i \mu +$$

$$+ 2 \sum_j \sum_i (l_j(a_j)\, l_i d_i \mu + l_j(l_i(a_j)\, d_i \mu) + l_i d_{ij}^2 \mu)\,.$$

Now estimate $|v|\,(x\colon \partial_v Q(x) \leqq \varepsilon)$, $v = d_v^k \mu$. Take a topological complement Y to L, denote by σ the image of $|v|$ under the natural projection on Y and choose smooth conditional measures v^y on $L + y$. We can supply L with the standard Lebesgue measure m using the natural isomorphism $J\colon L \leftrightarrow R^r$, $a_i \leftrightarrow e_i$. The slices $L + y$ are provided with the measures m_{-y}. By Lemma 2.3 $v^y = f^y m_{-y}$ and f^y can be chosen with $\sup_z |f^y(z)) \leqq \|d^v_{a_1 \ldots a_r} v\|$. Let $C = \{x \in X\colon \partial_v Q(x) \leqq \varepsilon\}$. It is easy to show that $m_{-y}(c \cap (L + y)) \leqq m(c \cap L)$ for all y. Denote by $M(Q, L)$ the minimum of $\partial_v Q$ on the unit sphere in L (determined by the isomorphism J). Then $m(C) \leqq c(r)\,.$ $.\, M(Q, L)^{-r/2}\, \varepsilon^{r/2}$. Hence we have

$$|v|\,(C) \leqq c(r)\, M(Q, L)^{-r/2}\, \|d^r_{a_1 \ldots a_r} v\|\, \varepsilon^{r/2}\,.$$

From these observations we can conclude (integrating by parts as in Theorem 2.1 that for each fixed $\alpha \in R^n$, $\|\alpha\| = 1$:

$$|\varphi(t\alpha)| \leqq c_k(\alpha)\, |t|^{-k}\,.$$

Moreover, the constant $c_k(\alpha)$ depends only on k, norms of $\partial_v^q Q$ in $L^2(d_v^e \mu)$ and $M(Q, L)$. From continuity arguments it is clear that for every $(\alpha_1, \ldots, \alpha_n) = \alpha$ in the unit sphere in R^n there exists a neighbourhood $V(\alpha)$ such that vectors $a_1, \ldots, a_k \in D$ can be chosen common and forms $\partial_v(s_1 Q_1 + \ldots + s_n Q_n)$ are uniformly nondegenerate provided $(s_1, \ldots, s_n) \in V(\alpha)$. The compactness of the sphere in R^n gives us the desired uniform estimates of φ.

Remark 3.1. The same arguments are valid for general polynomials of the second order.

Consider a stochastic differential equation

$$(3.3) \qquad d\xi_t = A(t, \xi_t) \circ dw_t + B(t, \xi_t)\, dt\,, \quad \xi_0 = 0\,, \quad t \in [0, T]\,,$$

where w is a standard d-dimensional Wiener process,

$$A\colon [0, T] \times R^d \to L(R^d)\,, \quad B\colon [0, T] \times R^d \to R^d$$

are C_b^∞-mappings (see [9], [16] for basic definitions; here stochastic differentials can be understood no matter in the sense of Ito or in the sense of Stratonovich). Real-valued symmetric function $K\colon [0, T] \times [0, T] \to R^1$ will be called nondegenerate if for each n it is possible to choose such $t_1, \ldots, t_n \in [0, T]$ that functions $k_1(\cdot) = K(\cdot, t_1), \ldots, k_n(\cdot) = K(\cdot, t_n)$ are linearly independent on $[0, T]$. Operator-valued mapping $K\colon [0, T] \times [0, T] \to L(R^d)$ with $K(t, s) \equiv K(s, t)$ will be called nondegenerate if for every n there exist $t_1, \ldots, t_n \in [0, T]$ such that for all $s_1, \ldots, s_n \in R^d$ with $\sum \|s_i\| = 1$ the map $s \mapsto \sum K(s, t_i)\, s_i$ is not identically zero.

Theorem 3.3. Suppose that $\inf(\det A) > 0$, $K\colon [0, T] \times [0, T] \to L(R^d)$ is a symmetric nondegenerate Holder mapping, ξ_t is the solution of the SDE (3.3), $Q(\omega) = \int_0^T \int_0^T (K(s, t)\, \xi_s(\omega), \xi_t(\omega))\, ds\, dt$. Then Q has infinitely differentiable distribution and its density belongs to $S(R^1)$.

Proof. The details of the full proof are too hard to be presented here, so we sketch only the main steps in the case $d = 1$. First we notice that the measure μ on $C_0[0, T]$ which corresponds to the solution η_t of (3.3) with $A \equiv 1$ is infinitely differentiable along subspace W_1 (defined before Theorem 2.2) and all its derivatives are absolute continuous with respect to μ and the Wiener measure W while their densities with respect to μ and W belong to $\bigcap_p L^p(\mu)$ and $\bigcap_p L^p(W)$ correspondently. It can be deduced from the Girsanov's theorem according to which $\mu \sim W$, $\mu = \varrho W$ where ϱ has an explicite form, simple for investigations (see [14], [16], [17]). Let $g(t, z) = \int_0^z A^{-1}(t, y)\, dy$, $f(t, \cdot)$ be the inverse diffeomorphism to $g(t, \cdot)$. Ito's formula (see [16])

$$dg(t, \xi_t) = dw_t + \left[\frac{\partial g}{\partial t}(t, \xi_t) + B(t, \xi_t)/A(t, \xi_t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, \xi_t)\, A(t, \xi_t)^2\right] dt\,.$$

Hence $\xi_t = f(t, \eta_t)$ where η_t is the solution of SDE with $A = 1$. Thus we have to study the functional

$$F(x) = \int_0^T \int_0^T K(s, t)\, f(t, x(t))\, f(s, x(s))\, dt\, ds$$

defined on $X = C_0[0, T]$ with the smooth measure μ corresponding to η_t. Follow the scheme of the proof of Theorem 2.3. Direct calculations show that for all h

$$\partial_h F(x) = 2 \int_0^T A(t, f(t, x(t)))\, h(t) \left(\int_0^T K(s, t)\, f(s, x(s))\, ds\right) dt\,.$$

As above we have to obtain estimates

$$\mu(x\colon \sup_{h \in S_m} (\partial_h F(x))^2 \leqq \varepsilon) = o(\varepsilon^k)\,.$$

It was shown in the proof of Theorem 2.3 that for this aim it suffices to establish the estimates

$$\mu(x\colon \sup_t |\int_0^T K(s,t) f(s, x(s))\, ds| \leqq \varepsilon) = o(\varepsilon^k)\,.$$

It can be done in the following way. Fix n and prove that the mapping $\Psi\colon x \mapsto \mapsto (\int_0^T K(s, t_1) f(s, x(s))\, ds, \ldots, \int_0^T K(s, t_n) f(s, x(s))\, ds)$ induces a measure on R^n with bounded density. Here t_i are chosen so that $K(\cdot, t_1), \ldots, K(\cdot, t_n)$ are linearly independent (this is the only place where we need the existence of such t_i). For proving the boundedness of the density of $\mu \circ \Psi^{-1}$ it suffices to have an estimate

$$\int \exp(i \sum \alpha_j \Psi_j(x))\, \mu(dx) = O(\|\alpha\|^{n-2})\,.$$

Now we have to resort to the method of Theorem 3.2. We omit the details, the full proof will be published elsewhere.

Corollary. For every k $\int \exp(it\, Q(x))\, \mu^\xi(dx) = o(t^{-k})$, $t \to \infty$, μ^ξ being a measure on $C_0[0, T]$, induced by ξ.

Remark 3.2. The same result is valid (with a simpler proof) for quadratic forms $Q(x) = \int_0^T (K(t)\, x(t), K(t)\, x(t))\, dt$, $K \in L(R^d)$, $\|\det K\|_{L^1} \neq 0$. In the one-dimensional case it is a consequence of Theorem 2.2. Certainly, this is the form of another type: it is non compact while the form considered in Theorem 3.3 is of Hilbert-Schmidt type (therefore the last one is more „degenerate" and more difficult to deal with). One might ask whether Theorem 3.3 is a consequence of Theorem 3.2. It would be the case if the measure μ^ξ were differentiable along some linear subspace. We have already mentioned that this is true when A is constant. In 1961 T. Pitcher [17] conjectured that if $d = 1$, $A(t, x) \equiv A(x) \neq$ const then whichever be a function $h \not\equiv 0$ on $[0, T]$ there does not exist any measure σ on $C[0, T]$ with $\mu^{\xi + \alpha h} \ll \sigma$ for all $\alpha \in R^1$, where $(\xi + \alpha h)_t = \xi_t + \alpha\, h(t)$. According to [1] it is the same as to say that μ^ξ can not be continuous in a nonzero direction. We shall prove Pitcher's hypothesis and in the same time give a negative answer to the question posed by Uglanov in [14].

Theorem 3.4. Denote by μ the measure on $C[0, T]$ induced by the stochastic process ξ_t satisfying the SDE (s. 3) where $d = 1$, A and B are Lipschitzian, $A(t, x) \equiv \equiv A(x)$ is not a constant, $A \in C^1(R)$, $A > 0$. Then μ has no nonzero directions of continuity.

Proof. We again present only a scheme of the proof. Denote by f the solution of the Cauchy problem $f'(t) = A(f(t))$, $f(0) = 0$. It is well-known that this solution exists on the whole line. The process $\eta_t = f(w_t)$ by the Ito's formula satisfies the equation

$$d\eta_t = A(\eta_t)\, dw_t + \tfrac{1}{2} A'(\eta_t)\, A(\eta_t)\, dt\,, \quad \eta_0 = 0\,,$$

and by the Girsanov's theorem induces the measure $\nu \sim \mu$. So from the very beginning we can assume that $\xi_t = f(w_t)$. For $t \in [0, T]$ write

$$L_t\varphi = \limsup_{s\to 0+} |\varphi(t+s) - \varphi(t)|/\sqrt{(2s\log\log s)}\,.$$

It follows from the law of the iterated logarithm (see [16]) that for μ-almost all $x \in C[0, T]$

$$L_t x = A(x(t))\,.$$

Suppose that μ is continuous in the direction of h and for some t $h(t) > 0$. The crucial point of the proof is to get an estimate

$$\limsup_{s\to 0+} |h(t+s) - h(t)|/\sqrt{s} < \infty\,. \tag{3.4}$$

Suppose it is not valid. Then there exists a sequence $t_n \downarrow 0$ with $s_n = (h(t+t_n) - h(t))/\sqrt{t_n} \to \infty$ (or $s_n \to -\infty$). Let $\alpha_n = s_n^{-1}$. Then $\alpha_n \to 0$ and by the continuity of μ $\|\mu_{\alpha_n h} - \mu\| \to 0$ (see [1]). Hence

$$\sup_{l\in X^*} |(\exp(i\alpha_n\, l(h)) - 1)\,\tilde{\mu}(l)| = \sup_{l\in X^*} |\tilde{\mu}_{\alpha_n h}(l) - \tilde{\mu}(l)| \to 0\,. \tag{3.5}$$

Let $l_n \in X^*$, $l_n(x) = (x(t+t_n) - x(t))/\sqrt{t_n}$. By (3.5) $\sup_n |(\exp(i\alpha_n\, l_n(h)) - 1)\,\tilde{\mu}(l_n)| \to 0$ and hence $\tilde{\mu}(l_t) \to 0$ because $\alpha_n\, l_n(h) \equiv 1$. On the other hand direct calculations (which are omited) show that

$$\lim_{n\to\infty} \tilde{\mu}(l_n) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\tfrac{1}{2}A(f(z))^2 - \tfrac{1}{2}z^2/t\right) \mathrm{d}z > 0\,.$$

This is a contradiction and so (3.4) holds. Hence $L_t h = 0$. Take a segment $I = [s - 2\delta,\ s + 2\delta]$ without zeros of A'. Suppose $\inf_I A' = m > 0$. Let $E = \{x\colon L_t x = A(x(t)),\ x(t) \in (s - \delta,\ s + \delta)\}$. It is easy to show that $\mu(E) > 0$ and from the continuity of μ we have: $E \cap (E - \lambda h) \neq \emptyset$, $|\lambda| \leqq r(E)$. Find some x in this intersection. Since $L_t h = 0$ and, therefore, $L_t(x + \lambda h) = L_t x$, we get the following equality: $A(x(t) + \lambda\, h(t)) = A(x(t))$. On the other hand for all $\alpha \in (0, \delta/h(t))$ $A(x(t) + \alpha\, h(t)) = A(x(t)) + \alpha\, h(t)\, A'(x(t) + \theta)$ where $\theta \in [0, \alpha\, h(t)]$, and consequently $A(x(t) + \alpha\, h(t)) > A(x(t))$. This is a contradiction.

Corollary. The conjecture of Pitcher [17] is true.

Remark 3.3. a) If a coefficient A depends on both variables t and x, the measure μ can possess nonzero directions of continuity or differentiability only when there exists an interval U such that $A(t, x) \equiv A(t, 0)$ for all $t \in V$. b) The analogous results are valid also for multidimensional diffusion processes, but in this case an $L(R^d)$-valued coefficient A should be considered nonconstant if for every nonconstant smooth $h\colon [0, T] \to R^d$ with $h(0) = 0$ the map $t \mapsto A(t, h(t))$ is nonconstant.

We conclude this section by noticing that Theorem 3.1 considerably strengthens the result of [18] where $X_1, \ldots, X_n$ were supposed to be one-dimensional and X was a Banach space. Theorem 3.2 for Hilbert spaces was also proved by Uglanov [18],

[19]. His methods differ very much from ours though he also uses the theory of differentiable measures. Integral-type functionals of the Brownian motion and related oscillatory integrals were studied in [20], [21]. We shall compare this approach with our one in § 5.

§ 4. Absolute continuity of distributions

Many assumptions on measures and functions can be weakened if we are interested only in absolute continuity of image-measures.

Proposition 4.1. Suppose that a Radon measure μ on an LCS X is continuous in directions $\{a_n\}$ and a Borel mapping $F: X \to R^d$ satisfies the following condition: for μ-almost every $x \in X$ it is possible to choose $v_1(x), \ldots, v_d(x) \in \{a_n\}$ such that the vectors $\omega_i(x) = \lim_{t \to 0} (F(x + tv_i(x)) - F(x))/t$, $i = 1, \ldots, d$, exist in R^d and are linearly independent. Then $\mu \circ F^{-1}$ has a density on R^d.

Proof. It is well-known that sets $D_i = \{x\colon \partial_{a_i} F(x) \text{ exists}\}$ are Borel and so the following sets are also Borel:

$$D(i_1, \ldots, i_d) = \{x \in D_{i_1} \cap \ldots \cap D_{i_d}\colon \partial_{i_j} F(x) \text{ are linearly independent}\} .$$

Since the measure $\mu|_{D(i_1, \ldots, i_d)}$ is also $\{a_n\}$-continuous, it suffices to examine the case when $v_1 = a_1, \ldots, v_d = a_d$ are common for μ-almost all x. By Lemma 2.1 this is reduced to the case $X = R^d$. So we have to prove the following lemma: if $E \subset R^d$ has positive Lebesgue measure and a Borel map $F: R^d \to R^d$ possesses linearly independent partial derivatives in every point of E then $F(E)$ is also a set of positive Lebesgue measure. For locally Lipschitzian F it is a direct consequence of the co-area formula of Theorem 3.2.5 in [22]. According to Theorem 3.1.4 in [22] F is approximately differentiable almost everywhere on E and by Theorem 3.1.8 [22] E can be covered (except some set of measure zero) by compact sets E_n on which F is Lipschitzian. It remains to notice that for almost all $x \in E_n$ partial derivatives $\partial_{a_i} F_n(x)$ are linearly independent (this fact follows from the condition and Fubini's theorem), F_n being any Lipschitz extensions of $F|_{E_n}$.

Proposition 4.1. considerably improves some analogous results obtained in [23], [24] where additional restrictions were imposed (in particular, mapping were assumed to have almost everywhere continuous Gateau derivatives and the choice of v_i was not arbitrary as in our case).

Remark 4.1. The mapping F in Proposition 4.1 can be assumed only μ-measurable if $\mu_{ta_n} \sim \mu$ for all t, n or if it is indicated that sets D_i are μ-measurable.

Proposition 4.2. Suppose that a Radon measure μ on an LCS X is differentiable in Skorokhod's sense in directions $\{a_i\}$ and a function $F: X \to R^1$ satisfies the following conditions: 1) F is the limit of the sequence of μ-measurable functions F_n converging in measure (or a.e.); 2) functions F_n are differentiable along $\{a_i\}$ and there exist such measurable functions g_i that for each compact set $K \subset X$: $\lim_{n\to\infty} \int_K |\partial_{a_i} F_n(x) - g_i(x)|\, |\mu|(\mathrm{d}x) = 0$; 3) the set of points where all g_i vanish is null for $|\mu|$. Then $\mu \circ F^{-1}$ has a density.

Proof. Let K be a compact in X and $i \in N$. We can assume that μ is nonnegative since $|\mu|$ is also differentiable in a_i (see [3]). As shown in [3] there exists a Borel function $\varphi: X \to [0, 1]$ which has a compact support S, equals to 1 on K and possesses the bounded derivative $\partial_{a_i}\varphi$. Then $\lambda = \varphi\mu$ is compactly supported and differentiable in Skorokhod's sense in a_i. It is clear that $\|\partial_i F_n - g_i\|_{L^1(\lambda)} \to 0$. For all $\psi \in C_0^\infty(R^1)$

$$\int \psi'(F(x))\, g_i(x)\, \lambda(\mathrm{d}x) = \lim_{n\to\infty} \int \psi'(F_n(x))\, \partial_{a_i} F_n(x)\, \lambda(\mathrm{d}x) =$$

$$= \lim_{n\to\infty} \int \partial_{a_i}(\psi \circ F_n)(x)\, \lambda(\mathrm{d}x)\,.$$

By (2.4) we have $\int \partial_{a_i}(\psi \circ F_n)(x)\, \lambda(\mathrm{d}x) = -\int \psi(F_n(x))\, d_{a_i} \lambda(\mathrm{d}x)$. Hence $|\int \psi'(F(x)) \,.\, g_i(x)\, \lambda(\mathrm{d}x)| \leqq \sup |\psi(s)|\, \|d_{a_i}\lambda\|$. This estimate in view of Lemma 2.2 signifies the absolute continuity of $(g_i\lambda) \circ F^{-1}$. Since K and i were arbitrary the same is true for $\mu \circ F^{-1}$.

This result strengthens the theorem of Davydov [25] in several directions (he considers Gaussian measures μ and F_n are supposed to be Frechet differentiable along the reproducing kernel of μ and absolutely continuous on lines $x + R^1 h$ for all $h \in D(\mu)$). In particular our results cover those of Shigekava [26], Nualart-Zakai [27].

Remark 4.2. The analogous results can be formulated and proved for functionals on abstract measurable manifolds.

Corollary. Let $X = C([0, T], R^d)$ or $X = L^2([0, T], R^d)$, μ be a measure on X induced by the solution ξ_t of SDE (3.3) with Lipschitzian A, B. Suppose that $A(\cdot, \cdot)$ is nondegenerate and a locally Lipschitzian map $F: X \to R^n$ satisfies the condition: $\mu(x \in X$: Gateau derivative $F'(x)$ exists and $F'(x)(X) \neq R^n) = 0$. Then $\mu \circ F^{-1}$ is absolutely continuous.

§ 5. Different approaches to the Malliavin calculus

Another way of studying distributions of functionals is presented in [28], [8], [29].

Definition 5.1. An Ornstein-Uhlenbeck-Malliavin operator on a probability space $(\Omega, \mathscr{B}, P)$ is a symmetric linear operator L in $L^2(P)$ defined on a dense linear subspace $\mathscr{R} \subset \bigcap_p L^p(P)$ and taking values in $\bigcap_p L^p(P)$ such that 1) for all $\varphi_1, \ldots, \varphi_n \in \mathscr{R}$ and $f \in C_b^\infty(R^n)$ the composition $f(\varphi_1, \ldots, \varphi_n)$ belongs to $\mathscr{R}$; 2) for all $\varphi \in \mathscr{R}$ $\Gamma(\varphi, \varphi) \geqq 0$

a.e., where $\Gamma(\varphi, \psi) = L(\varphi\psi) - \varphi L\psi - \psi L\varphi$; 3) for all $\varphi_1, \ldots, \varphi_n \in \mathscr{R}$, $f \in C_b^\infty(R^n)$

$$L(f(\varphi_1, \ldots, \varphi_n)) = \sum_i \frac{\partial f}{\partial x_i}(\varphi_1, \ldots, \varphi_n)\, L\varphi_i + \tfrac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i\, \partial x_j}(\varphi_1, \ldots, \varphi_n)\, \Gamma(\varphi_i, \varphi_j).$$

In [8] analogous operators were defined as generators of symmetric diffusion semigroups on $L^2(P)$. For convenience Definition 5.1 differs in nonessential details from that one given in [29].

Proposition 5.1. Let L be an operator of Ornstein-Uhlenbeck-Malliavin and $f_1, \ldots, f_n \in \mathscr{R}$. Then: 1) $(\Omega, \mathscr{B}, \mathscr{R} \cap L^\infty)$ is a measurable manifold in the sense of § 2; 2) mappings $\varphi \mapsto \Gamma(f_i, \varphi)$ on $\mathscr{R}$ are defining vector fields on Ω denoted by v_i (in particular the Malliavin matrix $\Gamma(f_i, f_j)$ coincides with $\partial v_i f_j$); 3) the measure P is differentiable along v_i and $d_{v_i}P = (-2Lf_i)\, P$.

The proof consists of simple calculations.

Proposition 5.2. Suppose that a measure μ on a measurable manifold $(X, \mathscr{B}, \mathscr{E})$ is differentiable along vector fields $v_1, \ldots, v_n$, $d_{v_i}\mu = \varrho_i\mu$, $\varrho_i \in \bigcap_p L^p(\mu)$ and the set $\mathscr{R} = \{\varphi \in \mathscr{E}\colon \partial_{v_i}\varphi,\ \partial^2_{v_i}\varphi \in \bigcap_p L^p(\mu)\}$ is dense in $L^2(\mu)$. Then the formula

$$L\varphi = \sum_{i=1}^n (\partial^2_{v_i}\varphi + \varrho_i\, \partial v_i \varphi)$$

defines on $\mathscr{R}$ an Ornstein-Uhlenbeck-Malliavin operator.

The proof is not difficult.

The Malliavin calculus is a branch of differential calculus for functions and measures on infinite dimensional spaces. In more narrow sense it is a method of studying differential properties of distributions of functionals of random processes (especially Gaussian) which is based on the integration by parts formula in infinite dimensions. Originally it was created by P. Malliavin for investigating transition probabilities for diffusion processes ξ_t (in other words, measures on R^d induced by the simplest map $\omega \mapsto \xi_t(\omega)$, t being fixed). An interesting discussion of this method (including different approaches) is presented in [30]. Proposition 5.1, 5.2 show that these approaches are equivalent at the abstract level (for example, compare our Theorem 2.1 with analogous results in [8], [9], [29]–[33]). Some diversities are explained by various choices of vector fields along which measures are differentiable. Besides, different authors apply abstract theorems to different concrete classes of measures and functionals. For instance, Smorodina [31] deals with stable measures. We have discussed differentiable measures. The theory of differentiable measures was suggested by S. Fomin (1966) and developed in [34], [2], [4], [35], [1] (relevant ideas belong also to Pitcher [17]). For the connection of the topics being discussed with stochastic calculus see [27], [36].

In the conclusion I would like to thank the Organizing Committe of the 17-th Winter School on Abstract Analysis for the invitation to participate and Dr. D. Preiss for useful discussions and valuable suggestions concerning our § 4. I am particularly indebted to Dr. L. Zajíček for kind attention.

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