Acta Universitatis Carolinae. Mathematica et Physica

Marek Balcerzak

On some ideals and related algebras of sets in the plane

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 2, 3--7

Persistent URL: http://dml.cz/dmlcz/701788

On Some Ideals and Related Algebras of Sets in the Plane

MAREK BALCERZAK

Łodź*)

Received 15 March 1989

We consider two σ-algebras (KL) and (LK) of sets in the plane, associated with the respective mixed measure-category Fubini products of ideals. We show that each set from (KL) (resp. (LK)) is contained in a special simple set from (KL) (resp. (LK) such that the difference of the sets is small.

Let $I = [0, 1]$ and let μ denote the Lebesgue measure on I. By $\mathbb{K}$ and $\mathbb{L}$ we denote, respectively, the σ-ideals of meager sets and Lebesgue null sets in I. For $x \in I$ and $B \subseteq I^2$, we write

$$B[\{x\}] = \{x \in I: \langle x, y\rangle \in B\}.$$

Consider the following Fubini-type product

$$\mathbb{K} \otimes \mathbb{L} = \{A \subseteq I^2: \text{there is a Borel } B \supseteq A \text{ such that}$$
$$\{x \in I : B[\{x\}] \notin \mathbb{K}\} \in \mathbb{L}\}$$

and let $\mathbb{L} \otimes \mathbb{K}$ be defined analogously. Then $\mathbb{K} \otimes \mathbb{L}$ and $\mathbb{L} \otimes \mathbb{K}$ are σ-ideals. There are some interesting results on them (see [M], [CKP], [CP], [G], [F1], [F2]).

Let us denote

(LM) – the family of Lebesgue measurable sets in I^2,
(BP) – the family of sets in I^2 with the Baire property,
(KL) – the σ-algebra generated by Borel sets in I^2 and by sets from $\mathbb{K} \otimes \mathbb{L}$,
(LK) – the σ-algebra generated by Borel sets in I^2 and by sets from $\mathbb{L} \otimes \mathbb{K}$.

The aim of the paper is to present some properties of (KL) and (LK).

The first proposition is a consequence of the known results obtained in [M], [G] and [F1].

Proposition 1. We have:

(a) $(KL)\setminus(LK) \neq \emptyset$, $(LK)\setminus(KL) \neq \emptyset$,
(b) $(LM) \cap (BP)\setminus((KL) \cup (LK)) \neq \emptyset$,

*) Institute of Mathematics, University of Łodź, ul. Stefana Banacha, 90-238 Łodź, Poland

(c) $(KL) \cap (LK) \setminus ((LM) \cup (BP)) \neq \emptyset$,
(d) (KL) and (LK) are invariant under Suslin's operation,
(e) (KL) and (LK) are not invariant under the symmetry with respect to the diagonal $y = x$.

Proof. Let A and B be disjoint Borel subsets of I such that $A \in \mathbb{K}$, $B \in \mathbb{L}$ and $A \cup B = I$ (cf. [0]). Let $H \subseteq I^2$ be a Bernstein set (i.e. $P \cap H \neq \emptyset$ and $P \setminus H \neq \emptyset$ for each perfect set P in I^2; cf. [0]). We have

$$H \cap (A \times I) \in (KL) \setminus (LK), \quad H \cap (B \times I) \in (LK) \setminus (KL),$$
$$H \cap ((A \times B) \cup (B \times A)) \in (LM) \cap (BP) \setminus ((KL) \cup (LK)),$$
$$H \cap ((A \times A) \cup (B \times B)) \in (KL) \cap (LK) \setminus ((LM) \cup (BP)),$$

which gives (a), (b) and (c). For instance, let us show the first of the above relations. Since $A \times I \in \mathbb{K} \otimes \mathbb{L}$, the condition $H \cap (A \times I) \in (KL)$ is obvious. Observe that $H \cap (A \times I) \notin \mathbb{L} \otimes \mathbb{K}$ since $H[\{x\}] \notin \mathbb{K}$ for all $x \in A$. Suppose that $H \cap (A \times I) \in (LK)$. Thus $H \cap (A \times I) = D \cup C$ for some Borel set D and for $C \in \mathbb{L} \otimes \mathbb{K}$. Since D is not in $\mathbb{L} \otimes \mathbb{K}$, it must be uncountable and thus contains a perfect set (cf. [Ku], § 33, I). Hence $D \setminus H \neq \emptyset$, which is impossible.

A slight modification of the standard construction leads to the Bernstein set symmetric with respect to the diagonal $y = x$. Then we have

$$H \cap (A \times I) \in (KL) \quad \text{but} \quad H \cap (I \times A) \notin (KL)$$

and

$$H \cap (B \times I) \in (LK) \quad \text{but} \quad H \cap (I \times B) \notin (LK),$$

which gives (e).

To prove (d), recall that any disjoint subfamily of $(KL) \setminus \mathbb{K} \otimes \mathbb{L}$ is countable (see [G], [F1]). The standard argument (see e.g. Lemma 1H (b) in [F1]) shows that (KL) is a Marczewski algebra, so it is closed under Suslin's operation (see [Ku], § 11, VII). The proof for (LK) is analogous.

Remarks. (1) It follows from (d) that all analytic subsets of I^2 are in $(KL) \cap (LK)$.

(2) We can consider $\mathbb{K}$ and $\mathbb{L}$ in the whole line $\mathbb{R}$ and thus (KL) and (LK) are defined for $\mathbb{R}^2$. Then the equivalent form of (e) states that (KL) and (LK) are not invariant with respect to the rotation around the origin at the angle $\pi/2$. What about other rotations?

The following result is derived from [F2].

Proposition 2. If $B \in \mathbb{L} \otimes \mathbb{K}$, then there are a Lebesgue null set $C \subseteq I$ (of type G_δ) and an F_σ set $F \subseteq I^2$ with meager vertical sections, such that $B \subseteq (C \times I) \cup F$.

We shall prove a certain generalization.

Theorem 1. If $B \in (LK)$, then there are a Lebesgue null set $C \subseteq I$ (of type G_δ) and an F_σ set $F \subseteq I^2$ such that $B \subseteq (C \times I) \cup F$ and $(F \setminus B)[\{x\}]$ is meager for each $x \in I$.

Proof. It suffices to consider the case when B is Borel. Let $B^* = \{x \in I: B[\{x\}] \in \in \mathbb{K}\}$, $B_* = B \cap (B^* \times I)$ and $B_{**} = B \setminus B_*$. Since B^* is Borel (cf. [V]), we have $B_* \in \mathbb{L} \otimes \mathbb{K}$. Hence, by Proposition 2, there are a set $C_* \in \mathbb{L}$ (of type G_δ) and an F_σ set $F_* \subseteq I^2$ with meager vertical sections, such that $B_* \subseteq (C_* \times I) \cup F_*$. Now, consider B_{**}. Let $\{V_n: n \in \omega\}$ be a base of open sets in I. If $B[\{x\}]$ is nonmeager, there is V_n such that $V_n \setminus B[\{x\}]$ is meager. Thus we have $I \setminus B^* = \bigcup_{n\in\omega} A_n$ where $A_n = \{x \in I: V_n \setminus B[\{x\}] \in \mathbb{K}\}$ for $n \in \omega$. It is known that the sets A_n are Borel (cf. [V]). For each $n \in \omega$, choose an F_σ set $F_n \subseteq A_n$ such that $\mu(A_n \setminus F_n) = 0$. Put $F_{**} = \bigcup_{n\in\omega} (F_n \times V_n)$ and $C_{**} = (I \setminus B^*) \setminus \bigcup_{n\in\omega} F_n$. Observe that $C_{**} \in \mathbb{L}$ (it can be enlarged to a G_δ set from $\mathbb{L}$) and $(F_{**} \setminus B)[\{x\}] \in \mathbb{K}$ for each $x \in I$. Finally, consider $B \setminus F_{**}$. It belongs to $\mathbb{L} \otimes \mathbb{K}$ since it is Borel and all its vertical sections are meager. Hence, by Proposition 2, we can find the respective sets C_{***} and F_{***} such that $B \setminus F_{**} \subseteq (C_{***} \times I) \cup F_{***}$. Put $C = C_* \cup C_{**} \cup C_{***}$ and $F = F_* \cup F_{**} \cup \cup F_{***}$. Then the assertion holds.

The analogous theorem is true for (KL).

Theorem 2. If $B \in (KL)$, then there are a meager set $C \subseteq I$ (of type F_σ) and a G_δ set $A \subseteq I^2$ such that $B \subseteq (C \times I) \cup A$ and $\mu((A \setminus B)[\{x\}]) = 0$ for each $x \in I$.

The proof is based on the lemma.

Lemma. Let $B \subseteq I^2$ be a Borel set. The following property holds:

(*) for each $\varepsilon > 0$, there are a sequence $\langle U_n: n \in \omega \rangle$ of open sets in I, and a sequence $\langle G_n: n \in \omega \rangle$ of open sets in I^2 and a meager set $C \subseteq I$ of type F_σ, such such that if

$$A = \bigcup_{n\in\omega} ((U_n \times I) \cap G_n),$$

then $B \subseteq (C \times I) \cup A$ and $\mu((A \setminus B)[\{x\}]) < \varepsilon$ for each $x \in I \setminus C$.

Proof. It is enough to show three facts:

(1) property (*) holds for open sets $B \subseteq I$;

(2) property (*) holds for $B = \bigcup_{m\in\omega} B_m$ if it holds for Borel B_m's;

(3) property (*) holds for $B = \bigcap_{m\in\omega} B_m$ if it holds for Borel B_m's such that $B_{m+1} \subseteq B_m$.

To show (1), put $U_n = I$ and $G_n = B$ for all $n \in \omega$ and let $C = \emptyset$.

To show (2), consider any B_m and, for $\varepsilon/2^{m+1}$, choose – by (*) – the respective sequences $\langle U_{mn}: n \in \omega \rangle$, $\langle G_{mn}: n \in \omega \rangle$ and a set C_m. Put

$$U_n = \bigcup_{m\in\omega} U_{mn}, \quad G_n = \bigcup_{m\in\omega} G_{mn}$$

for $n \in \omega$ and let

$$C = \bigcup_{m\in\omega} C_m .$$

Now, it is clear that (*) holds for B.

To show (3), consider any B_m and, for $\varepsilon/2$, choose – by (*) – the respective sequences $\langle U_{mn}: n \in \omega \rangle$, $\langle G_{mn}: n \in \omega \rangle$ and a set C_m. Put

$$E_m = \{x \in I: \mu((B_m \setminus B)[\{x\}]) < \varepsilon/2\} .$$

We have

$$I = \bigcup_{m \in \omega} E_m .$$

The sets E_m are Borel (cf. [Ke]) and $E_m \subseteq E_{m+1}$ for $m \in \omega$. Let $D_0 = E_0$ and $D_m = E_m \setminus E_{m-1}$ for $m > 0$. For each D_m, there is an open U_m such that $D_m \Delta U_m$ (the symmetric difference) is meager. Of course,

$$I = \bigcup_{m \in \omega} U_m \cup \bigcup_{m \in \omega} (D_m \Delta U_m)$$

and

$$B \subseteq \bigcap_{m \in \omega} ((C_m \times I) \cup \bigcup_{n \in \omega} (U_{mn} \times I) \cap G_{mn}) .$$

Thus

$$B \subseteq (C \times I) \cup A$$

where

$$C = \bigcup_{m \in \omega} C_m \cup (D_m \Delta U_m)$$

and

$$A = \bigcup_{m \in \omega} \bigcup_{n \in \omega} (((U_m \cap U_{mn}) \times I) \cap G_{mn})$$

(the set C can be enlarged, if necessary, to an F_σ meager set). Let $x \in I \setminus C$. There is a unique $m \in \omega$ such that $x \in D_m \cap U_m \setminus C_m$. Thus we have

$$\mu((A \setminus B)[\{x\}]) = \mu((\bigcup_{n \in \omega} G_{mn} \setminus B)[\{x\}] =$$

$$= \mu((\bigcup_{n \in \omega} G_{mn} \setminus B_m)[\{x\}]) + \mu((B_m \setminus B)) < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

This ends the proof.

Proof of Theorem 2. It suffices to consider the case when B is Borel. We define

$$C = \bigcup_{n \in \omega} C_n \quad \text{and} \quad A = \bigcap_{n \in \omega} A_n \setminus C$$

where A_n and C_n are obtained by Lemma for $\varepsilon = 1/n$.

Corollary. If $B \in \mathbb{K} \otimes \mathbb{L}$, then there are a meager set $C \subseteq I$ (of type F_σ) and a G_δ set $A \subseteq I^2$ with vertical sections of measure zero, such that $B \subseteq (C \times I) \cup A$.

Observe that the converses to Theorems 1 and 2 are also true. Thus we have the characterizations of (KL) – and (LK) – measurability.

I would like to thank Professor D. Fremlin for helpful remarks and discussions.

References

[CKP] Cichoń J., Kamburelis A. and Pawlikowski J., On dense subsets of the measure algebra, Proc. Amer. Math. Soc. 94 (1985), 142—146.

[CP] Cichoń J., Pawlikowski J., On ideals of subsets of the plane and on Cohen reals, J. Symb. Logic 51 (1986), 560—569.

[F1] Fremlin D. H., Measure — additive coverings and measure selectors, Dissert. Math. 260 (1987).

[F2] Fremlin D. H., The partially ordered sets of measure theory and Tukey's ordering, 1988 (preprint).

[G] Gavalec M., Iterated products of ideals of Borel sets, Colloq. Math. 50 (1985), 39—52.

[Ke] Kechris A. S., Measure and category in effective descriptive set theory, Ann. Math. Logic 5 (1973), 337—384.

[Ku] Kuratowski K., Topologie I, PWN, Waszawa 1958.

[M] Mendez C. G., On sigma-ideals of sets, Proc. Math. Soc. 60 (1976), 124—128.

[O] Oxtoby J. C., Measure and Category, Springer-Verlag, New York 1971.

[V] Vaught R., Invariant sets in topology and logic, Fund. Math. 82 (1974), 269—293.