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MINI-COURSE: AN INTRODUCTION TO ℓ_2-HOMOLOGY

PETER ZVENGROWSKI

Dedicated to the Memory of Heiner Zieschang, 1936-2004

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CHAPTER I
INTRODUCTION

The inspiration for this mini-course comes from similar lectures given by Beno Eckmann during one of his visits to western Canada, and the author's subsequent attempts to understand this fascinating subject. In 2000 a substantial paper of Eckmann's appeared in the Israel J. Math [4], based on the notes (by Guido Mislin) from a mini-course he gave in 1997 at the Mathematical Research Institute, ETH Zürich. The present notes are completely based on these notes of Eckmann, with very little, if any, claim to originality. An introductory chapter (Chapter II) on basic Hilbert space theory has been added, since the subsequent material is completely based on this. Most but not all of the Eckmann paper is covered, however it is the author's hope that the present mini-course will give a fairly thorough introduction to the basics of the subject, so that anyone attending the mini-course should have no great difficulty in reading the remainder of [4] or other literature involving ℓ_2-homology.

The subject originated with Atiyah's ideas (cf. [1]), in 1976, of applying Hilbert space technique to algebraic topology to obtain refined (ℓ_2) invariants of (generally infinite) cell complexes. It has been used by many authors since then and become an increasingly important tool, notably the ideas of ℓ_2-homology (ℓ_2-cohomology) and the associated ℓ_2-Betti numbers β_i.

We shall now attempt to give some intuitive feeling for the subject and its application by means of three examples.

1.1.1 Example. A nice place to start any topological discussion is Euler's famous formula $V - E + F = 2$. In more modern language, for S^2 the 2-sphere, one says $\chi(S^2) = 2 = \alpha_0 - \alpha_1 + \alpha_2$, where χ is the Euler characteristic and α_i the number of i-cells. One also has the homology groups $H_0(S^2) \approx \mathbb{Z}$, $H_2(S^2) \approx \mathbb{Z}$, and otherwise $H_i(S^2) = 0$. The rank of the finitely generated abelian group H_i is called the i-th Betti number b_i of the space. It can also be defined by taking homology with coefficients in $\mathbb{Q}$ or $\mathbb{R}$, and taking the dimension of the resulting vector space. Thus $b_0(S^2) = \dim_{\mathbb{R}} H_0(S^2; \mathbb{R}) = 1$, $b_2(S^2) = \dim_{\mathbb{R}} H_2(S^2; \mathbb{R}) = 1$, and $b_i(S^2) = 0$ otherwise. This illustrates a well known theorem of algebraic topology,

$$\chi(X) = \sum_{i \geq 0} (-1)^i \alpha_i(X) = \sum_{i \geq 0} (-1)^i b_i(X),$$

for X a finite CW-complex. In the present case $\chi(S^2) = 2 = b_0 - b_1 + b_2 = 1 - 0 + 1$.

For S^1, $\chi(S^1) = 0$, $b_0(S^1) = b_1(S^1) = 1$ and $b_i(S^1) = 0$ otherwise. So here the theorem takes the form $\chi(S^1) = 0 = b_0 - b_1 = 1 - 1$.

1.1.2 Example. Now consider the familiar covering projection $p : \mathbb{R} \twoheadrightarrow S^1$, $p(t) = \exp(2\pi i t)$. Here S^1 is the unit circle in $\mathbb{C}$, with base point 1, and $\mathbb{R}$ has base point 0. Let $S^1 = e^0 \cup e^1$, a cellular decomposition with one 0-cell $e^0 = \{1\}$ and one 1-cell e^1. The corresponding cells of $\mathbb{R}$ can be written $\bar{e}^0_j = \{j\} \subset \mathbb{R}$, $j \in \mathbb{Z}$, and $\bar{e}^1_j = (j, j+1) \subset \mathbb{R}$. The fundamental group $\pi_1(S^1, 1) = \mathbb{Z}$ acts freely on $\mathbb{R}$ by translations, this action is cellular (permutes the cells) and free (only the identity $0 \in \mathbb{Z}$ fixes any cell). This is an example of a regular covering, i.e. $p_* \pi_1(\mathbb{R}, 0) = \pi_*\{e\} = \{e\}$, the trivial group, is a normal subgroup of $\pi_1(S^1, 1)$. It is also an example of a cocompact group action by $G = \mathbb{Z}$ on $Y = \mathbb{R}$, namely $X = Y/G = S^1$ is compact.

Now trying to define $\alpha_0, \alpha_1, \ldots$ here would be futile, since $\alpha_0(Y), \alpha_1(Y)$ are infinite. To define (real) homology of Y one starts with the chains $K_i(Y; \mathbb{R})$, the real vector space with basis the i-cells of Y. Thus an element of, say $K_1(Y; \mathbb{R})$ is a sum $\sum_{j \in \mathbb{Z}} r_j \bar{e}^1_j$, with $r_j \in \mathbb{R}$, almost all r_j equal 0. We could consider a more "global" chain by allowing infinite sums, i.e. remove the condition almost all $r_j = 0$. But this will create other problems, e.g. the formula for the boundary map may well have divergent sums. This can be overcome by considering the ℓ_2-chains, where we impose the condition of square summability $\sum_{j \in \mathbb{Z}} r_j^2 < \infty$.

A chain complex of Hilbert spaces is thus obtained, with extra structure as G-modules arising from the action of G. The resulting homology groups, modulo a few technical details, are the ℓ_2-homology groups. They are generally (countably) infinite dimensional Hilbert spaces, but they have another equivariant type dimension, called the von Neumann dimension β_i. It turns out that β_i is a non-negative real number, which we call the i-th ℓ_2-Betti number, and the Euler characteristic can be computed in terms of these. In the present example it turns out that $\beta_i(S^1) := \beta_i(\mathbb{R}, \mathbb{Z}) = 0$, $i \geq 0$, so one obtains the (not very exciting) formula

$$\chi(S^1) = 0 = \sum_{i \geq 0} (-1)^i \cdot 0 .$$

Of course, more interesting applications will appear later.

1.1.3 Example. In this example we illustrate an application to algebra. Let G be a finitely presented group (finitely many generators and finitely many relations). In general many different presentations are possible, the deficiency of any given presentation P is the number of generators g_P less the number of relations r_P, i.e. $g_P - r_P$. Since r_P can be increased at will without changing G, say by simply repeating the same relation, $g_P - r_P$ has no lower bound. However, by considering the abelianization G_{ab} of G it is easily seen (cf. 7.2.1) that $g_P - r_P \leq \operatorname{rank}(G_{ab}) := b_1(G)$, the first Betti number of G (or equivalently of the Eilenberg-MacLane space $K(G,1)$). Thus the deficiency of any presentation P has an upper bound, and we define $\operatorname{def}(G) := \max\{g_P - r_P : P \text{ is finite presentation of } G\}$. From the above, $\operatorname{def}(G) \leq b_1(G)$. The difficult part of finding $\operatorname{def}(G)$ will generally be finding as sharp an upper bound as possible, one can then hope to find a presentation P achieving this bound. Thus, theorems of the following type can be quite useful.

Theorem. $\operatorname{def}(G) \leq 1 + \beta_1(G)$ (*cf.* § 7.2).

For example, for the free group F_n on n letters, we shall show $\operatorname{def}(F_n) = n$. Similarly, for the fundamental group σ_g of an orientable surface of genus g, we shall show $\operatorname{def}(\sigma_g) = 2g - 1$. For F_n this can be proved using the usual Betti numbers or the ℓ_2-Betti numbers, however for σ_g the usual Betti numbers do not suffice whereas ℓ_2-Betti numbers do give the required upper bound.

Remark. In Example 1.1.2, although the number of cells α_0, α_1 for $Y = \mathbb{R}$ are infinite, one could argue that homologically $\mathbb{R}$ is quite simple, with $b_0 = 1$ and $b_j = 0$, $j > 0$. However, these (ordinary) Betti numbers are unrelated to the G action, and in slightly more complicated examples, the ordinary Betti numbers of Y, just like the α_i, can also be infinite.

Notation. Here are some frequently used notations in these notes.

$\subset$	proper subset
$\subseteq$	subset
e	the neutral (identity) element of a group G
$X^{(n)}$	the n-skeleton of a CW-complex
$\boxplus$	orthogonal internal direct sum decomposition

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CHAPTER II
HILBERT SPACE, A BRIEF REVIEW

This chapter gives a quick review of basic material on Hilbert space. It contains quite standard material (cf. [6], [11]), apart perhaps from Lemma 2.3.10, and can be safely omitted by analysts or anyone who has had a course on Hilbert space in the not too distant past.

2.1 Introduction to Hilbert space

2.1.1 Definition. An $\mathbb{R}$-vector space M together with a map $\langle\ ,\ \rangle : M \times M \to \mathbb{R}$ is an $\mathbb{R}$-inner product space if $\langle\ ,\ \rangle$ is bilinear, symmetric, and strictly positive.

We shall only need $\mathbb{R}$-vector spaces in these notes. The corresponding definition for $\mathbb{C}$-vector spaces, incorporating conjugation is standard.

2.1.2 Definition. For $x \in M$, $\|x\| = \langle x, x\rangle^{1/2}$ is the associated norm.

2.1.3 Definition. For $x, y \in M$, $x \perp y$ if and only if $\langle x, y\rangle = 0$. We say x, y are orthogonal.

2.1.4 Proposition. *The following are equivalent:*
(a) $x = 0$,
(b) $\langle x, y\rangle = 0$ *for all* $y \in M$,
(c) $x \perp y$ *for all* $y \in M$,
(d) $\|x\| = 0$.

2.1.5 Polarization identity. $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$.

2.1.6 Parallelogram law. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

2.1.7 Corollary (Pythagorean Theorem). *For* $x, y \in M$,

$$x \perp y \quad \textit{iff} \quad \|x \pm y\|^2 = \|x\|^2 + \|y\|^2 .$$

Proof. Since $x \perp y$, 2.1.5 shows that $\|x+y\| = \|x-y\|$, then use 2.1.6. □

2.1.8 Schwarz inequality. For $x, y \in M$, $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$.

Proof. For any $s, t \in \mathbf{R}$, one has

$$0 \leq \|sx + ty\|^2 = \begin{bmatrix} s & t \end{bmatrix} \cdot A \cdot \begin{bmatrix} s \\ t \end{bmatrix},$$

where

$$A = \begin{bmatrix} \|x\|^2 & \langle x, y\rangle \\ \langle y, x\rangle & \|y\|^2 \end{bmatrix}$$

(called the Gram matrix of x, y) is symmetric, positive semi-definite. So A has real eigenvalues $\lambda_1, \lambda_2 \geq 0$, whence $\det A = \lambda_1 \cdot \lambda_2 \geq 0$. Since $\det A = \|x\|^2\|y\|^2 - \langle x, y\rangle^2$, this gives the Schwarz inequality. □

Note that the same proof shows that the Gram matrix $A = [\langle x_i, x_j\rangle]$ of any n vectors $x_1, \ldots, x_n$ is symmetric positive semi-definite.

Using the Schwarz inequality (in 2.1.9 (c) below) we see that $\|x\|$ satisfies the axioms for a norm.

2.1.9 Proposition. *The axioms for a norm hold for $\|x\|$, namely*
(a) $\|x\| \geq 0$, $\|x\| = 0$ *iff* $x = 0$,
(b) $\|rx\| = |r| \cdot \|x\|$, $r \in \mathbb{R}$,
(c) *Triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$.

Proof of (c). $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$. □

2.1.10 Corollary. *Setting $d(x, y) = \|x - y\|$, (M, d) is a metric space. Furthermore, $\|x\| = d(x, 0)$ is continuous, and thus by 2.1.5, $\langle x, y\rangle$ also is continuous in x and in y.*

2.1.11 Definition.
(a) A Banach space is a normed vector space which is complete as a metric space.
(b) A Hilbert space is an inner product vector space which is complete as a metric space.

2.1.12 Remark. Not every norm comes from an inner product. In fact, it can be shown that a norm comes from an inner product iff the parallelogram law 2.1.6 is satisfied. For example, on $\mathbb{R}^2$, defining the norm of a vector $\boldsymbol{v} = (x, y)$ by $\|\boldsymbol{v}\| = |x| + |y|$ will give a norm for which the parallelogram law fails.

2.1.13 Examples of Hilbert spaces.
(a) $\mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$.
(b) $\ell_2 = \ell_2(\mathbb{N}) = \{(x_1, x_2, \ldots) : x_i \in \mathbb{R}, \sum_{i=1}^\infty x_i^2 < \infty\}$, and $\langle x, y\rangle = \sum x_i y_i$ which is absolutely convergent since, by the Schwarz inequality,

$$\sum_{i=1}^n |x_i y_i| \leq \Big(\sum_{i=1}^n x_i^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{i=1}^n y_i^2\Big)^{\frac{1}{2}} \leq \Big(\sum_{i=1}^\infty x_i^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{i=1}^\infty y_i^2\Big)^{\frac{1}{2}} = \|x\| \cdot \|y\|$$

hence

$$\sum_{i=1}^\infty |x_i y_i| \leq \|x\| \cdot \|y\| < \infty.$$

(c) The product of two (or more generally any finite number) Hilbert spaces $M \times N$ with

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \,,$$

or equivalently

$$\|(x,y)\|^2 = \|x\|^2 + \|y\|^2 \,.$$

(d) The collection of holomorphic (complex) functions $f(z)$ on the interior of the unit disc, with $|f(z)|^2$ integrable with respect to planar Lebesgue measure, an important example of a complex Hilbert space.

2.1.14 Remarks.
(a) $\mathbb{R}^\infty = \{(x_1, x_2, \ldots, 0, 0, \ldots) : \text{ almost all } x_i \text{ equal } 0\}$ is a dense linear subspace of ℓ_2.
(b) The completeness of ℓ_2 is the Riesz-Fischer theorem, a proof is given in Chapter II Appendix A.
(c) Everything done in this chapter will also work in complex Hilbert spaces, with minor modifications due to the conjugation in $\mathbb{C}$, e.g. the polarization identity is

$$\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)\,.$$

2.2 Orthogonality

2.2.1 Definition. A subset of a Hilbert space M that is closed under addition and scalar multiplication is called a linear subspace. If it is also closed topologically then it is called a Hilbert subspace, which we often write (at least in this chapter) H-subspace. In the latter case the H-subspace is obviously itself a Hilbert space with the same norm as M.

2.2.2 Theorem. *Let Y be an H-subspace of M, $x \in M$, and*

$$\delta = \inf\{\|y - x\| : y \in Y\}\,.$$

Then there exists a unique $y_0 \in Y$ such that $\delta = d(x, y_0)$, and $(x - y_0) \perp Y$.

Proof. Uniqueness is clear since if y_1 is another such element then

$$(x - y_1) \perp Y \Rightarrow y_1 - y_0 = (x - y_0) - (x - y_1) \perp Y\,.$$

But $(y_1 - y_0) \in Y$, hence $y_1 - y_0 = 0$. For existence of y_0, let $\{y_n\} \subset M$ with $\|y_n - x\| \to \delta$. From the Parallelogram law,

$$2\|y_n - x\|^2 + 2\|y_m - x\|^2 = \|y_m + y_n - 2x\|^2 + \|y_n - y_m\|^2\,.$$

Since $\frac{(y_m + y_n)}{2} \in M$,

$$\|y_m + y_n - 2x\|^2 = 4\|\frac{y_m + y_n}{2} - x\|^2 \geq 4\delta^2\,,$$

thus

$$\|y_n - y_m\|^2 \leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\delta^2\,.$$

As $m, n \to \infty$ the right hand side approaches $2\delta^2 + 2\delta^2 - 4\delta^2 = 0$ which implies that $\{y_n\}$ is Cauchy. Since Y is an H-subspace, $y_n \to y_0 \in M$. By continuity (2.1.10),

$$\|y_0 - x\| = \lim_{n\to\infty} \|y_n - x\| = \delta .$$

Finally, let $t \in \mathbb{R}$, $y \in Y$, and put $z = y_0 - x$. Then

$$\|z + t\langle y, z\rangle y\| = \|(y_0 + t\langle y, z\rangle y) - x\| \geq \delta = \|z\|$$

implies that

$$0 \leq 2t\langle y, z\rangle^2 + t^2\langle y, z\rangle^2\|y\|^2$$

for all $t \in \mathbb{R}$. Taking t sufficiently small negative implies that $\langle y, z\rangle = 0$. □

2.2.3 Corollary. *Let $X \subset Y \subseteq M$ be a proper inclusion of H-subspaces. Then there exists $y \in Y$, $y \neq 0$, with $y \perp X$. This is true even if Y is just a linear subspace.*

2.2.4 Definition. For any subset $A \subset M$, $A^\perp = \{x : x \perp A\}$.

Note that $A^\perp$ is always an H-subspace since $\langle\ ,\ \rangle$ is continuous by 2.1.10 and since $A^\perp$ is clearly a linear subspace.

2.2.5 Theorem. *Let $X \subseteq M$ be any H-subspace, then*

$$M = X \boxplus X^\perp \quad (\textit{internal orthogonal direct sum}).$$

Proof. Suppose $x \in M$. By 2.2.2 there exists a unique $y_0 \in X$ such that $x - y_0 \in X^\perp$. Then $x = y_0 + (x - y_0)$ implies that $M = X + X^\perp$. Clearly,

$$X \cap X^\perp = \{0\},$$

so this is an internal orthogonal direct sum. □

Of course, this is only interesting when $X \subset M$.

2.2.6 Definition. With the notation of 2.2.5, define the orthogonal projection $\pi_X : M \twoheadrightarrow X$ by $\pi_X(x) = y_0$.

2.2.7 Proposition.
(a) *π_X is linear,*
(b) *$\pi_X|_X = \mathrm{id}_X$,*
(c) *π_X is idempotent, i.e., $\pi_X \circ \pi_X = \pi_X$.*

2.2.8 Proposition. *For any subset $A \subset M$, $A^\perp = (\bar{A})^\perp$.*

Proof. $A \subseteq \bar{A} \Rightarrow (\bar{A})^\perp \subseteq A^\perp$, and the reverse implication is immediate from the continuity of $\langle\ ,\ \rangle$. □

2.3 Bounded linear operators

2.3.1 Definition. A linear transformation $f : M \to N$ of Hilbert spaces is called a bounded operator if there exists $\kappa \geq 0$ such that $\|f(x)\| \leq \kappa\|x\|$ for all $x \in M$. In this case, the infimum of all such κ is defined to be $\|f\|$.

2.3.2 Definition. For any linear transformation $f : M \to N$,

$$\operatorname{Ker} f = \{x \in M : f(x) = 0\} \subseteq M$$

and

$$\operatorname{Im} f = \{y \in N : y = g(x) \text{ for some } x \in M\} \subseteq N\,.$$

2.3.3 Proposition. *Let $f : M \to N$ be a bounded operator. Then*
(a) *f is continuous,*
(b) *the bounded linear operators and Hilbert spaces form a category $\mathfrak{H}$,*
(c) $\operatorname{Ker} f$ *is an H-subspace,*
(d) $\operatorname{Im} f$ *is a linear subspace.*

Proof. (a): $\|f(y) - f(x)\| = \|f(y - x)\| \leq \kappa \cdot \|y - x\|$, where $\kappa = \|f\|$, clearly implies continuity.
(b): $\|\mathrm{id}_M\| = 1$, $\|g \circ f\| \leq \|g\| \cdot \|f\|$ suffices to show that id_M is bounded and the composition of two bounded operators is again bounded, i.e. $\mathfrak{H}$ is a category.
(c) and (d): Trivial. □

The fact that $\operatorname{Im} f$ is not in general closed (cf. Example 2.3.8 (a) below) will be of great importance in the later development of the theory.

2.3.4 Definition. If $f : M \to \mathbb{R}$ is a bounded operator, then it is called a bounded linear functional (on M). The collection of all such f is written as M^*, the dual space of M.

2.3.5 Riesz representation theorem. *If $f \in M^*$, then there exists a unique $y \in M$ such that $f(x) = \langle x, y\rangle$ for all $x \in M$.*

Proof. If $\operatorname{Ker} f = M$ then $f = 0$ and $y = 0$ works, so assume that $\operatorname{Ker} f \subset M$. Since it is an H-subspace, by Corollary 2.2.3, there exists $y_1 \in (\operatorname{Ker} f)^{\perp}$, $y_1 \neq 0$, say without loss of generality, $\|y_1\| = 1$. Let $f(y_1) = a \neq 0$ (since $y_1 \notin \operatorname{Ker} f$) and set $y = ay_1$. Then $\|y\| = |a| > 0$ and $f(y) = af(y_1) = a^2 > 0$. For any $x \in M$,

$$f\Big(x - \frac{f(x)}{a^2} \cdot y\Big) = f(x) - \frac{f(x)}{a^2} \cdot a^2 = 0\,,$$

hence $x - \frac{f(x)}{a^2} y \in \operatorname{Ker} f \perp y$ and

$$\langle x, y\rangle = \Big\langle \Big(x - \frac{f(x)}{a^2} y + \frac{f(x)}{a^2} y\Big), y\Big\rangle = 0 + \frac{f(x)}{a^2}\langle y, y\rangle = f(x)\,. \quad \square$$

2.3.6 Definition. Let $\phi : M_1 \to M_2$ be a bounded operator and set $f(x) = \langle \phi x, y\rangle$, where $y \in M_2$. It is easy to see that f is a bounded linear functional, so by the Riesz Representation Theorem 2.3.5, $f(x) = \langle x, \phi^* y\rangle$ for a unique $\phi^* y \in M_1$. Clearly, $\phi^* : M_2 \to M_1$ is linear and ϕ^* is called the adjoint of ϕ.

2.3.7 Proposition.
(a) *ϕ^* is a bounded operator, with $\|\phi^*\| = \|\phi\|$,*
(b) *$\phi^{**} = \phi$,*
(c) *$(a\phi + b\psi)^* = a\phi^* + b\psi^*$,*
(d) *if ϕ is invertible then $(\phi^{-1})^* = (\phi^*)^{-1}$,*
(e) $\operatorname{Ker} \phi^* = (\operatorname{Im} \phi)^{\perp}$ *(which also is equal to $\overline{\operatorname{Im} \phi}^{\perp}$ by* 2.2.8).

2.3.8 Examples.
(a) Let $f : \ell_2 \to \ell_2$, $f(x_1, x_2, \ldots) = (\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$, then $\|f\| = 1$, and $\mathbb{R}^\infty \subset$

$\operatorname{Im} f \Rightarrow \operatorname{Im} f$ is dense. But $(1, \frac{1}{2}, \frac{1}{3}, \ldots) \notin \operatorname{Im} f$, so $\operatorname{Im} f$ is not an H-subspace. Here, $\operatorname{Ker} f = \{0\}$.
(b) $\pi_X : M \twoheadrightarrow X$ is a bounded operator with $\|\pi_X\| = 1$ (assuming $X \neq \{0\}$), $\operatorname{Im} \pi_X = X$, $\operatorname{Ker} \pi_X = X^\perp$.
(c) Among all linear transformations $f : \ell_2 \to \ell_2$, a cardinality argument can be used to show that "most" are unbounded. But exhibiting a specific unbounded operator seems difficult.

2.3.9 Lemma. *Let X be a linear subspace of M. Then X is dense iff $X^\perp = \{0\}$ $(= (\bar{X})^\perp$ from 2.2.8).*

Proof. $\Rightarrow$ Let $x \in X^\perp$ with $\|x\| = 1$, supposing $X^\perp \neq \{0\}$. Since X is dense, there exists $y \in X$ with $\|y - x\| < \frac{1}{3}$. But then $x \perp y \Rightarrow \|y - x\|^2 = \|y\|^2 + \|x\|^2 > 1$, a contradiction.
$\Leftarrow$ Since $M = \bar{X} \boxplus (\bar{X})^\perp$ by 2.2.5, and $(\bar{X})^\perp = \{0\}$, we have $M = \bar{X}$, i.e. X is dense. □

The next lemma is not generally included in the basics of Hilbert space, but it will be needed in Chapter III.

2.3.10 Lemma. *If $f : M_1 \rightarrowtail M_2$ is an injective bounded operator, then $\operatorname{Im}(f^*f)$ is dense in M_1.*

Proof. First note that $\operatorname{Im}(f^*f)$ is a linear subspace of M_1. Let $y \in M_1$, $y \perp \operatorname{Im}(f^*f)$. Then $0 = \langle f^*fx, y\rangle = \langle fx, fy\rangle$ for all $x \in M_1$. In particular, $\langle f^*fy, y\rangle = 0 \Rightarrow 0 = \langle f(y), f(y)\rangle = \|f(y)\|^2 \Rightarrow f(y) = 0$ by 2.1.4 $\Rightarrow y = 0$ as f is injective. By Lemma 2.3.9, $\operatorname{Im}(f^*f)$ is dense. □

2.3.11 Definition.
(a) $f : M \to M$ is self adjoint (symmetric) if $f = f^*$.
(b) $f : M \to M$ is orthogonal if f is invertible and $f^{-1} = f^*$. Orthogonality is easily seen to be equivalent to $\|fx\| = \|x\|$ for all $x \in M_1$, i.e. f is norm preserving (and hence also inner product preserving, i.e. $\langle fx, fy\rangle = \langle x, y\rangle$ for all $x, y \in M$).

At this stage in a course on Hilbert space, one would soon turn to the study of eigenvalues of a bounded operator f, the collection of eigenvalues being called the spectrum of f (a subset of $\mathbb{C}$), and the spectral theorem for bounded operators that are self adjoint, or more generally that are normal ($ff^* = f^*f$). This would take us too far afield, but we will need the following special case of the spectral theorem. A self adjoint operator f always has real eigenvalues and is called **positive definite** if $\langle fx, x\rangle > 0$ for all $x \neq 0$, or equivalently all eigenvalues are positive real numbers; similarly **positive semi-definite**. The next theorem is essentially the same as that in [11], p.265.

2.3.12 Theorem. *Every positive (semi-)definite bounded operator A possesses a unique positive (semi-)definite square root, denoted $A^{1/2}$. It can be represented as the limit (in the strong sense) of a sequence of polynomials in A, and hence commutes with any bounded transformation that commutes with A.*

2.3.13 Remark. We have omitted the discussion of a Hilbert basis for a Hilbert space M, as distinct from an algebraic basis. The idea is clear, e.g. $e_1 = (1, 0, 0, \ldots), e_2 =$

$(0,1,0,0,\ldots),\ldots$ form a Hilbert basis for ℓ_2 since their linear span is dense in ℓ_2. Thus $\dim_{\mathfrak{H}}\ell_2 = \aleph_0$. In these notes we will only be concerned with separable Hilbert space M, i.e. $\dim M \le \aleph_0$. A bounded operator is determined by its values on a Hilbert basis, by continuity.

Appendix A

Proof of the Riesz-Fischer theorem. In any metric space, a convergent sequence is Cauchy, so it remains to prove that, conversely, any Cauchy sequence $\{x_n\}$, $x_n = (x_{n1}, x_{n2}, \ldots) \in \ell_2$, converges in ℓ_2. For any fixed i,

$$|x_{ni} - x_{mi}| \le \|x_n - x_m\|$$

implies that the sequence $x_{1i}, x_{2i}, \ldots$ is a Cauchy sequence in $\mathbb{R}$. So it converges, say to $u_i = \lim_{n\to\infty} x_{ni} \in \mathbb{R}$.

Choose M so that $m, n \ge M \Rightarrow \|x_n - x_m\| \le \frac{1}{2}$. Then $n \ge M$ implies that

$$\|x_n\| = \|x_M + (x_n - x_M)\| \le \|x_M\| + \frac{1}{2}. \tag{1}$$

Next write $x_n^{(k)} = (x_{n_1}, \ldots, x_{n_k}), u^{(k)} = (u_1, \ldots, u_k)$. Since $\lim_{n\to\infty} x_{n_i} = u_i$, i.e. $\lim_{n\to\infty} |x_{n_i} - u_i| = 0$, one has $\lim_{n\to\infty} \|x_n^{(k)} - u^{(k)}\| = 0$. So there exists N_k with $\|x_n^{(k)} - u^{(k)}\| < \frac{1}{2}$, $n \ge N_k$. Then $n \ge N_k$ implies that

$$\|u^{(k)}\| = \|x_n^{(k)} + (u^{(k)} - x_n^{(k)})\| < \|x_n^{(k)}\| + \frac{1}{2} \le \|x_n\| + \frac{1}{2},$$

that is

$$n \ge N_k \Rightarrow \|u^{(k)}\| < \|x_n\| + \frac{1}{2}. \tag{2}$$

Combining (1) and (2), for $n \ge \max\{M, N_k\}$, we have

$$\|u^{(k)}\| < \|x_M\| + 1.$$

Since M does not depend on k, this shows that $\|u^{(k)}\|$ is bounded above, i.e. $u \in \ell_2$.

Finally, choose L so that $m, n \ge L \Rightarrow \|x_n - x_m\| < \frac{\epsilon}{3}$, for a given $\epsilon > 0$. Also, it is clearly possible to choose k large enough so that $\|x_L - x_L^{(k)}\| < \frac{\epsilon}{6}$, $\|u - u^{(k)}\| < \frac{\epsilon}{6}$. We also have $\|x_n^{(k)} - x_L^{(k)}\| \le \|x_n - x_L\| < \frac{\epsilon}{3}$ for all $n \ge L$, whence $\|u^{(k)} - x_L^{(k)}\| \le \frac{\epsilon}{3}$ by taking the limit as $n \to \infty$. Combining, we have $n \ge L$ implies that

$$\|u - x_n\| \le \|u - u^{(k)}\| + \|u^{(k)} - x_L^{(k)}\| + \|x_L^{(k)} - x_L\| + \|x_L - x_n\|,$$

i.e.

$$\|u - x_n\| < \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{3} = \epsilon.$$

This shows $x_n \to u$. □

CHAPTER III
HILBERT G-MODULES AND VON NEUMANN DIMENSION

In this chapter, most of the algebraic machinery needed for our study of ℓ_2-homology is introduced. The main extra ingredient is a group G that acts freely and isometrically on a Hilbert space. In the topological applications G will turn out to be a quotient group of the fundamental group $\pi_1(X)$, where X is a finite CW-complex. Such a group is always countable, and this will be assumed henceforth, although it is possible to define $\ell_2 G$ even for G uncountable.

3.1 Definition of $\ell_2 G$ and its module structure

3.1.1 Definition. Let G be a countable group, then

$$\ell_2 G = \left\{ f : G \xrightarrow{f} \mathbb{R},\ \sum_{x \in G} (f(x))^2 < \infty \right\}.$$

For uncountable groups G one would simply use all functions $f : G \to \mathbb{R}$ with "countable support", but for our purposes countable groups suffice. To simplify notation, $\sum$ will denote $\sum_{x\in G}$ unless otherwise noted, and we write $f = \sum f(x)x$. Then $\langle f, g\rangle = \sum f(x)g(x)$, which is absolutely convergent just as in Example 2.1.13 (b), and the notation $\ell_2 G$ is consistent with $\ell_2\mathbb{N}$ as used in that example.

3.1.2 Definition. The group algebra $\mathbb{R}[G] = \mathbb{R}G \subseteq \ell_2 G$ consists of those functions with finite support, i.e.

$$\mathbb{R}G = \{ r : G \xrightarrow{f} \mathbb{R},\quad r(x) = 0 \text{ for almost all } x \in G \}.$$

This is clearly a dense linear subspace of $\ell_2 G$ (if G is finite then $\mathbb{R}G = \ell_2 G$) and just like the group ring $\mathbb{Z}G$ it has a multiplication that turns it into the "group algebra",

$$rs = \Big(\sum_{x\in G} r(x)x\Big) \cdot \Big(\sum_{y\in G} s(y)y\Big) = \sum_{x,y\in G} r(x)s(y)xy\,,$$

all sums being finite (we will generally use letters $r, s \in \mathbb{R}G$). Letting $xy = z$, we can rewrite the above equation as a "convolution product"

$$rs = \sum_{z\in G}\Big(\sum_{y\in G} r(zy^{-1})s(y)\Big)z = \sum_{z\in G}\Big(\sum_{x\in G} r(x)s(x^{-1}z)\Big)z\,.$$

3.1.3 Observation. The algebra $\mathbb{R}G$ is associative, has unity $1 = 1 \cdot e$, and is commutative iff G is commutative. Alas, the multiplication in $\mathbb{R}G$ does not extend to $\ell_2 G \supseteq \mathbb{R}G$ in general, there are convergence problems as Example 3.1.4 below explicitly shows. However, there is no problem to multiply elements of $\ell_2 G$ (on the left or on the right) by elements of $\mathbb{R}G$ (again, since elements of $\mathbb{R}G$ are *finite* sums). So the correct way to think of $\ell_2 G$ is as an $\mathbb{R}G$ bimodule, and we write $r \cdot f$ ($f \cdot r$) for the product of an element of $\mathbb{R}G$ with one of $\ell_2 G$ (or vice versa). For $r \in \mathbb{R}G$, we write $L_r : \ell_2 G \to \ell_2 G$ for left multiplication by r, similarly for R_r.

3.1.4 Example. Take $G = \mathbb{Z}$, and define $f, g : G \to \mathbb{R}$ by

$$f(n) = \begin{cases} 0, & n \le 0 \\ n^{-3/5}, & n \ge 1 \end{cases}, \qquad g(n) = f(-n).$$

Note $\sum_{n \ge 1} n^{-6/5}$ is convergent, so $f, g \in \ell_2 G$. However $f \cdot g \notin \ell_2 G$, indeed

$$(f \cdot g)(n) > \int_{n+1}^{\infty} \frac{dx}{[x(x-n)]^{3/5}} > \int_{n+1}^{\infty} \frac{dx}{x^{6/5}} = \frac{5}{(n+1)^{1/5}},$$

showing that $f \cdot g$ is not square summable.

3.1.5 Proposition.
(a) *For $r, s \in \mathbb{R}G$, $r \cdot s = rs$ (left or right action),*
(b) *G acts by isometries, i.e. for $r = 1 \cdot y$, both L_y and R_y are isometries of $\ell_2 G$,*
(c) *For $r, s \in \mathbb{R}G$ and $f \in \ell_2 G$, $r \cdot (s \cdot f) = (r \cdot s) \cdot f$, $(f \cdot s) \cdot r = f \cdot (s \cdot r)$.*

Proof. (a) This holds by definition.
(b) Obvious since $\{f(y^{-1}x) : x \in G\} = \{f(x) : x \in G\}$.
(c) $y \cdot (z \cdot \sum f(x)x) = y \cdot \sum f(z^{-1}x)x = y \cdot \sum g(x)x$, where $g(x) := f(z^{-1}x)$. Thus, we get

$$y \cdot \left(z \cdot \sum f(x)x\right) = \sum g(y^{-1}x)x = \sum f(z^{-1}y^{-1}x)x = \sum f\left((yz)^{-1}x\right)x,$$

i.e. $y \cdot (z \cdot \sum f(x)x) = (yz) \cdot \sum f(x)x$ holds for $y, z \in G$, and extends by linearity to all $r, s \in \mathbb{R}G$. □

3.1.6 Proposition. *The action of $\mathbb{R}G$ on $\ell_2 G$ (left or right) is by bounded operators.*

Proof. Let $r = \sum r(y)y \in \mathbb{R}G$, $f \in \ell_2 G$. Using 3.1.5 (b), we have $\|y \cdot f\| = \|f\|$, thus

$$\|r \cdot f\| = \|\sum r(y)y \cdot f\| \le \sum \|r(y)y \cdot f\| = \sum |r(y)|\,\|y \cdot f\| = \sum |r(y)|\,\|f\|$$

which implies that $\|L_r\| \le |r| := \sum |r(y)| < \infty$, being a finite sum. □

Notice that $(\ell_2 G)^n$ also becomes an $\mathbb{R}G$ bimodule with bounded operators, where $r \cdot (f_1, \ldots, f_n) = (r \cdot f_1, \ldots, r \cdot f_n)$, the so-called diagonal action. One frequently says G-module or left G-module, omitting the $\mathbb{R}$, similarly for G-invariant and G-equivariant. We close this section with four useful technical results.

3.1.7 Proposition. *Let M be any Hilbert space that is a (left) $\mathbb{R}G$-module with G acting by isometries (e.g. $M = (\ell_2 G)^n$). If V is a G-invariant Hilbert subspace, then so is $V^{\perp}$.*

Proof. Since $V^{\perp}$ is a Hilbert subspace, it suffices to show $v \in V^{\perp} \Rightarrow y \cdot v \in V^{\perp}$, for $y \in G$. But for any $w \in V$,

$$\langle w, y \cdot v \rangle = \langle y \cdot y^{-1} \cdot w, y \cdot v \rangle = \langle y^{-1} \cdot w, v \rangle = 0$$

since $y^{-1} \cdot w \in V$. □

3.1.8 Corollary. *If V is a G-invariant Hilbert subspace, then π_V is G-equivariant.*

Proof. Let $f = v + w \in M$ where $v \in V, w \in V^{\perp}$ (uniquely). For any $y \in G$, $y \cdot f = y \cdot v + y \cdot w$ with $y \cdot v \in V$ and (by 3.1.7) $y \cdot w \in V^{\perp}$. Then by definition

$$\pi_V(y \cdot f) = y \cdot v = y \cdot \pi_V(f).$$

□

Note that $\mathbb{R}G \otimes_{\mathbb{R}G} \ell_2 G \simeq \ell_2 G$, so that $(\mathbb{R}G)^n \otimes_{\mathbb{R}G} \ell_2 G \simeq (\ell_2 G)^n$ since tensor product distributes over direct sums. We then have

3.1.9 Proposition. *Let $\phi : (\mathbb{R}G)^n \to (\mathbb{R}G)^m$ be a morphism of (right) $\mathbb{R}G$-modules. Then the induced operator*

$$\tilde{\phi} := \phi \otimes_{\mathbb{R}G} \ell_2 G : (\ell_2 G)^n \to (\ell_2 G)^m$$

is bounded. Similarly for left $\mathbb{R}G$-modules with $\tilde{\psi} := \ell_2 G \otimes_{\mathbb{R}G} \psi$.

Proof. The $m \times n$ matrix $[\phi_{ij}]$ of ϕ satisfies

$$\phi(a_1, \ldots, a_n) = \Big(\sum_{j=1}^{n} \phi_{ij} a_j, \ldots, \sum_{j=1}^{n} \phi_{mj} a_j \Big) \quad a_j, \phi_{ij} \in \mathbb{R}G .$$

Write $\phi_{ij} = \sum t_{ij}(x) x$ (finite sum) and $|\phi_{ij}| := \sum |t_{ij}(x)|$. Then for $f \in \ell_2 G$, one has $\|\phi_{ij} \cdot f\| \leq |\phi_{ij}| \|f\|$ just as in 3.1.6, so

$$\|\tilde{\phi}(f_1, \ldots, f_n)\|^2 = \sum_i \Big\| \sum_j (\phi_{ij} \cdot f_j) \Big\|^2 \leq \sum_{i,j} |\phi_{ij}|^2 \|f_j\|^2 ,$$

i.e.

$$\|\tilde{\phi}(f_1, \ldots, f_n)\|^2 \leq \sum_{i,j} |\phi_{ij}|^2 \|(f_1, \ldots, f_n)\|^2 = \Big(\sum_{i,j} |\phi_{ij}|^2 \Big) \|(f_1, \ldots, f_n)\|^2 . \qquad \square$$

3.1.10 Proposition. *If $f : M_1 \to M_2$ is a G-equivariant bounded operator of Hilbert $\mathbb{R}G$-modules on which G acts by isometries, then*
(a) *so is $f^* : M_2 \to M_1$,*
(b) *so is $g : M_1 \to M_1$, where $g^2 = f^* f$, g is self adjoint and positive definite.*

Proof. (a) Let $\alpha \in G$, $y \in M_2$. For any $x \in M_1$, $\langle x, f^*(\alpha \cdot y) \rangle = \langle f(x), \alpha \cdot y \rangle = \langle \alpha^{-1} \cdot f(x), (\alpha^{-1}\alpha) \cdot y \rangle = \langle f(\alpha^{-1} \cdot x), y \rangle = \langle \alpha^{-1} \cdot x, f^*(y) \rangle = \langle x, \alpha \cdot f^*(y) \rangle$. Thus $\langle x, f^*(\alpha \cdot y) - \alpha \cdot f^*(y) \rangle = 0$, so by 2.1.4, $f^*(\alpha \cdot y) = \alpha \cdot f^*(y)$.
(b) $\langle x, f^* f x \rangle = \langle fx, fx \rangle \geq 0$ implies that $f^* f$ is self adjoint and positive semi-definite. So, g exists by Theorem 2.3.12. From (a), $g^2 = f^* f$ is G-equivariant and again, by Theorem 2.3.12, the same is true for g. $\square$

3.2 Hilbert G-modules

3.2.1 Definition. (a) A Hilbert G-module M is a left $\mathbb{R}G$-module M which is a Hilbert space on which G acts by isometries such that M is isometrically G-isomorphic to a G-invariant subspace of $(\ell_2 G)^n$, for some n.
(b) Morphisms of Hilbert G-modules are the bounded G-equivariant operators $f : M_1 \to M_2$, and this forms a category $\mathfrak{H}G$.

3.2.2 Quotient modules. Let M be a Hilbert G-module and $V \subset M$ a G-invariant linear subspace. Then $\bar{V}$ is G-invariant and $M/\bar{V}$ has a natural Hilbert space structure with

$$\| [w] \| = \inf\{ \|\tilde{w}\| : \pi_{\bar{V}}(\tilde{w}) = w \} , \quad w \in M .$$

Furthermore, $\pi_{\bar{V}}$ induces, by restriction to $V^\perp$, a G-equivariant isometric isomorphism of Hilbert G-modules $V^\perp \xrightarrow{\approx} M/\bar{V}$.

3.2.3 Definition. A map $f : M_1 \to M_2$ of Hilbert G-modules is a
(a) weak isomorphism if f is injective, bounded, G-equivariant, $\operatorname{Im} f$ is dense in M_2,
(b) strong isomorphism if f is an isometric G-equivariant isomorphism of Hilbert spaces.

The next theorem is a little surprising, and quite useful.

3.2.4 Theorem. *If $f : M_1 \to M_2$ is a weak isomorphism, then there exists a strong isomorphism $h : M_1 \to M_2$.*

Proof. As in the proof of 3.1.10 (b), f^*f is self adjoint positive semi definite, indeed positive definite since f is injective. It also has $\operatorname{Im}(f^*f)$ dense by Lemma 2.3.10. So, as in 3.1.10, there exists a positive definite self adjoint operator g with $g^2 = f^*f$, and (also by 3.1.10) g is G-equivariant. Furthermore, $\operatorname{Im} g \supseteq \operatorname{Im} g^2 = \operatorname{Im}(f^*f)$ and hence is dense. Since g is injective, $g^{-1} : \operatorname{Im} g \to M_1$, which is bijective, exists; we set $\tilde{h} = f \circ g^{-1} : \operatorname{Im} g \to M_2$. Then $\operatorname{Im} \tilde{h} = \operatorname{Im} f$ is dense in M_2, and using $g^* = g$, for any $x, y \in \operatorname{Im} g$, we have

$$\langle \tilde{h}x, \tilde{h}y \rangle = \langle fg^{-1}(x), fg^{-1}(y) \rangle = \langle f^*fg^{-1}(x), g^{-1}(y) \rangle = \langle g^2g^{-1}(x), g^{-1}(y) \rangle ,$$

i.e.

$$\langle \tilde{h}x, \tilde{h}y \rangle = \langle gg^{-1}(x), g^*g^{-1}(y) \rangle = \langle x, gg^{-1}(y) \rangle = \langle x, y \rangle .$$

Hence $\tilde{h} : \operatorname{Im} g \to \operatorname{Im} f$ is an isometric isomorphism, and since $\operatorname{Im} g \subseteq M_1$, $\operatorname{Im} f \subseteq M_2$ are dense, $\tilde{h}$ extends by continuity to an isometric isomorphism $h : M_1 \to M_2$. Since f and g are already known to be G-equivariant, so are (successively) g^{-1}, $\tilde{h}$, h and thus h is a strong isomorphism of Hilbert G-modules. □

3.2.5 Definition. Two Hilbert G-modules M_1 and M_2 are isomorphic ($M_1 \approx M_2$) if there exists a weak isomorphism $M_1 \to M_2$.

By Theorem 3.2.4, the existence of a weak isomorphism implies the existence of a strong isomorphism, thus this is an equivalence relation. As a second application, we have the following:

3.2.6 Proposition. *Let $\phi : M_1 \to M_2$ be a bounded G-equivariant operator of Hilbert G-modules. Then* $(\operatorname{Ker}\phi)^\perp \approx M_1/\operatorname{Ker}\phi \approx \overline{\operatorname{Im}\phi}$.

Proof. The first isomorphism follows from 3.2.2. The composition $i \circ \rho$ where $i : \operatorname{Im}\phi \hookrightarrow \overline{\operatorname{Im}\phi}$ is the inclusion map, and $\rho : M_1/\operatorname{Ker}\phi \to \operatorname{Im}\phi$ is the standard bijective map, is a weak isomorphism, so the second isomorphism follows from Theorem 3.2.4. □

3.3 Von Neumann Dimension

3.3.1. Our goal in this section is to define an "equivariant" dimension, called the von Neumann dimension, $\dim_G M$, of a Hilbert G-module M satisfying
(a) $\dim_G M \in \mathbb{R}_o^+$,
(b) $\dim_G M = 0$ iff $M = 0$,
(c) $\dim_G M = \dim_G N$ if $M \approx N$,
(d) $\dim_G(M \oplus N) = \dim_G M + \dim_G N$,
(e) $M \subseteq N \Rightarrow \dim_G M \leq \dim_G N$,

(f) $\dim_G(\ell_2 G) = 1$,
(g) G finite $\Rightarrow \dim_G M = \frac{1}{|G|}\dim_{\mathbb{R}} M$,
(h) If H is a subgroup of G with finite index then $\dim_G M = \dim_H M/[G:H]$.

3.3.2 Remarks. (a) This idea goes back to the 1936 paper of Murray and von Neumann [9], and is closely related to what they call the centre-valued trace.
(b) For G finite, both (f) and (h) follow from (g).
(c) From (d) and (f), $\dim_G(\ell_2 G)^n = n$.
(d) For $G = \{e\}$, $\ell_2 G = \mathbb{R}$ and $\dim_G M = \dim_{\mathbb{R}} M$, i.e. the theory reduces to ordinary linear algebra.

3.3.3 Definition. The Kaplansky trace map $\rho : \mathbb{R}G \to \mathbb{R}$ is given by $\rho(\sum r(x)x) = r(e)$.

3.3.4 Definition. The von Neumann algebra $N(G) = \hom_{\mathfrak{H}G}(\ell_2 G, \ell_2 G)$ is the algebra of bounded left G-equivariant operators $\ell_2 G \to \ell_2 G$.

3.3.5 Definition. Conjugation on $\ell_2 G$ (or $\mathbb{R}G$ by restriction) is the map $f = \sum f(x)x \longmapsto \bar{f} = \sum f(x)x^{-1}$.

Now recall $\ell_2 G$ is an $\mathbb{R}G$-bimodule. The right action of $\mathbb{R}G$ on $\ell_2 G$ is by bounded left G-equivariant operators (check that $L_y R_z = R_z L_y \in N(G)$, this means that R_z is left G-equivariant), so in this sense $\mathbb{R}G \subseteq N(G)$ as a subalgebra. The adjoint map $\phi \longmapsto \phi^*$ gives an involution on $N(G)$ which turns it into a real C^*-algebra. In $\ell_2 G$ the adjoint of R_y is easily seen to be $R_{y^{-1}}$, this shows that under the inclusion $\mathbb{R}G \subseteq NG$, conjugation in $\mathbb{R}G$, as defined in 3.3.5, corresponds to the adjoint in NG.

We now extend the Kaplansky trace map to a trace on NG, as follows.

3.3.6 Definition. Let $\phi \in NG$, then $\mathrm{tr}_G(\phi) := \langle \phi(e), e\rangle \in \mathbb{R}$.

3.3.7 Proposition.
(a) *If* $\phi \in \mathbb{R}G$ *then* $\mathrm{tr}_G(\phi) = \rho(\phi)$,
(b) $\mathrm{tr}_G(\phi) = \mathrm{tr}_G(\phi^*)$, *where* $\phi \in NG$.

Proof. (a) Set $\phi = \sum r(x)x$, then $\mathrm{tr}_G(\phi) = \langle e \cdot \sum r(x)x, e\rangle = \langle \sum r(x)x, e\rangle = r(e)\cdot 1 = r(e)$.
(b) $\mathrm{tr}_G(\phi) = \langle \phi(e), e\rangle = \langle e, \phi^*(e)\rangle = \mathrm{tr}_G(\phi^*)$. □

The definition of the von Neumann dimension will now be briefly indicated; establishing then all the properties given in 3.3.1 is not difficult but will be omitted here to remain within the time constraints of the mini-course.

3.3.8 Definition. (a) Let $M_n(N(G))$ be the algebra of bounded left G-equivariant operators $(\ell_2 G)^n \to (\ell_2 G)^n$ (thus, $M_1(N(G)) = N(G)$). Any operator $F \in M_n(N(G))$ gives rise in the usual way to an $n \times n$ matrix $[F_{ij}]$, where each $F_{ij} \in N(G)$. Then

$$\mathrm{tr}_G(F) := \sum_{i=1}^{n} \mathrm{tr}_G(F_{ii}).$$

(b) Let $V \subseteq (\ell_2 G)^n$ be a G-invariant Hilbert subspace. By Corollary 3.1.8, $\pi_V \in M_n(N(G))$. Then

$$\dim_G V := \mathrm{tr}_G(\pi_V) \in \mathbb{R}.$$

3.3.9 Definition. Let M be an arbitrary Hilbert G-module and choose a G-equivariant isometric isomorphism $\alpha : M \xrightarrow{\approx} V \subseteq (\ell_2 G)^n$. Then

$$\dim_G(M) := \dim_G V .$$

Of course one should check this defintion is well defined, i.e. independent of the choices of n and α. Again, this is not difficult but is omitted here.

3.3.10 Remark. It is a classical result that $\mathbb{Z}[G]$ has no idempotents apart from $0, 1$. However, in $\mathbb{R}[G]$, where say $G = C_2 = \{1, t\}$, $r = \frac{1}{2}(1 + t)$ is idempotent.

Kaplansky Conjecture. If G is torsion free then $\mathbb{R}[G]$ has no non-trivial idempotents.

In 1972 Zalesskii [12] showed $\mathrm{tr}_G(e) \in \mathbb{Q}$ for e idempotent in $\mathbb{R}[G]$ and G torsion free. Strengthening this from $\mathbb{Q}$ to $\mathbb{Z}$ would prove the conjecture. Further work in this direction was done in 1998 by Burger and Valette [2].

CHAPTER IV
REAL HOMOLOGY OF FINITE COMPLEXES AND HARMONIC CHAINS

In this chapter the ordinary homology of a finite CW-complexes with real coefficients, $H_*(X; \mathbb{R})$, is considered from a slightly novel point of view, using harmonic chains. This approach will make the introduction of ℓ_2-homology, in Chapter V, relatively straightforward. As a "preview", consider the usual short exact sequence defining the i-th homology groups of X,

$$0 \to B_i \hookrightarrow Z_i \to H_i \to 0 .$$

With real coefficients this becomes

$$0 \to B_i(X; \mathbb{R}) \hookrightarrow Z_i(X; \mathbb{R}) \to H_i(X; \mathbb{R}) \to 0 ,$$

a short exact sequence of vector spaces which necessarily splits, i.e. (supressing the $\mathbb{R}$ in the notation) $Z_i \approx B_i \oplus H_i$ as an external direct sum. Then also $Z_i = B_i \boxplus (B_i)^\perp$ as an internal orthogonal direct sum, with $B_i^\perp \approx H_i$. We shall study $B_i^\perp := \mathcal{H}_i$, the so-called harmonic chains.

4.1 Harmonic chains

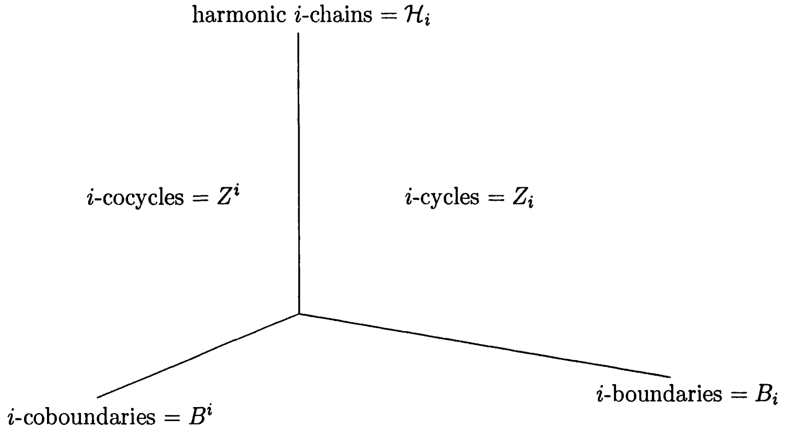
Let X be a finite CW-complex with (integral) cellular chain complex $(K_*(X), d)$. Its real chain complex is then $C_* := \mathbb{R} \otimes K_*(X)$, with $1 \otimes d$ as differential, also written d. Let $\sigma_1, \ldots, \sigma_{\alpha_i}$ be the i-cells of X, then these form the natural basis for $C_i(X)$ (which we often write simply C_i). Thus

4.1.1 Proposition. *$C_i(X)$ is a finite dimensional real Hilbert space of dimension α_i, with orthonormal basis $\{\sigma_1, \ldots, \sigma_{\alpha_i}\}$ and associated inner product $\langle\ ,\ \rangle : C_i \times C_i \to \mathbb{R}$.*

4.1.2 Definition. $\delta_{i-1} = d_i^* : C_{i-1}(X) \to C_i(X)$.

Thus, $\langle \delta_{i-1} x, y \rangle = \langle x, d_i y \rangle$ (also $\langle \delta_i x, y \rangle = \langle x, d_{i+1} y \rangle$). This is equivalent to the next result.

4.1.3 Proposition. $\mathrm{Ker}\, d_i = (\mathrm{Im}\, \delta_{i-1})^\perp$, *and* $\mathrm{Ker}\, \delta_i = (\mathrm{Im}\, d_{i+1})^\perp$.



4.1.4 Definition. $Z_i = \operatorname{Ker} d_i$, $B_i = \operatorname{Im} d_{i+1}$, $Z^i = \operatorname{Ker} \delta_i$, $B^i = \operatorname{Im} \delta_{i-1}$, all subspaces of C_i. We call Z_i, B_i, Z^i, B^i respectively the i-cycles, i-boundaries, i-cocycles, i-coboundaries.

4.1.5 Proposition.
(a) $B_i \subseteq Z_i$, $\quad B^i \subseteq Z^i$,
(b) $C_i = B^i \boxplus Z_i = B_i \boxplus Z^i$,
(c) $B^i \perp B_i$.

Proof. (a) It follows from $d_i d_{i+1} = 0$ and hence also $0 = (d_{i-1}d_i)^* = d_i^* d_{i-1}^* = \delta_i \delta_{i-1}$.
(b) From Theorem 2.2.5, $C_i = \operatorname{Im} \delta_{i-1} \boxplus (\operatorname{Im} \delta_{i-1})^\perp$, and using 4.1.3, 4.1.4 $C_i = B^i \boxplus Z_i$. Similarly for $B_i \boxplus Z^i$.
(c) It follows from $\langle \delta_{i-1}x, d_{i+1}y\rangle = \langle x, d_i d_{i+1} y\rangle = 0$ for all $x, y \in C_i$. ☐

4.1.6 Corollary. $C_i = B_i \boxplus B^i \boxplus (Z_i \cap Z^i)$.

Proof. $C_i = B_i \boxplus Z^i = B_i \boxplus (Z^i \cap C_i) = B_i \boxplus (Z^i \cap (B^i \boxplus Z_i)) = B_i \boxplus (Z^i \cap B^i) \boxplus (Z^i \cap Z_i) = B_i \boxplus B^i \boxplus (Z_i \cap Z^i)$. ☐

4.1.7 Definition. The harmonic i-chains of X are

$$\mathcal{H}_i(X) := Z_i(X) \cap Z^i(X).$$

Thus, $C_i = B_i \boxplus B^i \boxplus \mathcal{H}_i$, called the Hodge-de Rham decomposition of $C_i(X)$. The following diagram is a useful mnemonic for this orthogonal decomposition of $C_i(X)$.

4.1.8 Definition. The Laplacian is $\Delta_i := d_{i+1}\delta_i + \delta_{i-1}d_i : C_i \to C_i$.

4.1.9 Proposition. $\mathcal{H}_i = \operatorname{Ker} \Delta_i$.

Proof. $\mathcal{H}_i \subseteq \operatorname{Ker} \Delta_i$ is clear. Conversely, suppose $\Delta_i x = 0$, then $d_{i+1}\delta_i(x) = -\delta_{i-1}d_i(x)$ in $B_i \cap B^i = \{0\}$. This implies that $d_{i+1}\delta_i(x) = \delta_{i-1}d_i(x) = 0 \Rightarrow \delta_i(x) \in B^{i+1} \cap Z_{i+1} = \{0\}$ and $d_i(x) \in B_{i-1} \cap Z^{i-1} = \{0\}$. So $x \in \operatorname{Ker} \delta_i \cap \operatorname{Ker} d_i = \mathcal{H}_i$. ☐

4.2 Euler characteristic and Morse inequalities

4.2.1 Definition.

$$\chi(X) = \sum_{i \geq 0} (-1)^i \alpha_i$$

is the (usual) Euler characteristic of X.

4.2.2 Definition. $b_i(X) = \dim_{\mathbb{R}} \mathcal{H}_i(X)$ is the (not quite usual) i-th Betti number of X.

In fact, the usual definition of Betti number is $\dim_{\mathbb{R}} Z_i - \dim_{\mathbb{R}} B_i$.

4.2.3 Proposition. $b_i(X) = \dim_{\mathbb{R}} Z_i - \dim_{\mathbb{R}} B_i$.

Proof. Similar to the proof of 4.1.6, we have $Z_i = Z_i \cap C_i = Z_i \cap (B_i \boxplus Z^i) = B_i \boxplus \mathcal{H}_i$ implies that $\dim_{\mathbb{R}} Z_i = \dim_{\mathbb{R}} B_i + \dim_{\mathbb{R}} \mathcal{H}_i$. □

Thus $b_i(X)$ equals in fact the usual Betti number, implying the following theorem found in every algebraic topology text.

4.2.4 Theorem. $\chi(X) = \sum_{i \geq 0} (-1)^i b_i(X)$.

Similarly we have, with essentially the same proof

4.2.5 Morse inequalities. For any $k \geq 0$,

$$\alpha_k - \alpha_{k-1} + \alpha_{k-2} - \ldots + (-1)^k \alpha_0 \geq b_k - b_{k-1} + \ldots + (-1)^k b_0 .$$

Proof. Consider $C_{i-1} = B_{i-1} \boxplus Z^{i-1} \overset{\delta_i}{\twoheadrightarrow} B^i$. Since $\operatorname{Ker} \delta_i = Z^{i-1}$, $\delta_i : B_{i-1} \overset{\approx}{\to} B^i$, so $\dim_{\mathbb{R}} B^i = \dim_{\mathbb{R}} B_{i-1}$. Now $C_i = B_i \boxplus B^i \boxplus \mathcal{H}_i$ implies that $\alpha_i = \dim B_i + \dim B_{i-1} + b_i$, i.e. $(\alpha_i - b_i) = \dim B_i + \dim B_{i-1}$. Then

$$\sum_{i=0}^{k} (-1)^{k-i} (\alpha_i - b_i) = (\dim B_k + \dim B_{k-1}) - (\dim B_{k-1} + \dim B_{k-2}) + \ldots$$

or $\quad \sum_{i=0}^{k} (-1)^{k-i} (\alpha_i - b_i) = \dim B_k \geq 0.$ □

Notice that by taking $k > \dim K$ this also proves 4.2.4.

4.3 Homology and cohomology

4.3.1. We have already noted in the proof of 4.2.3 that $Z_i = B_i \boxplus \mathcal{H}_i$. Since $H_i(X; \mathbb{R}) = Z_i / B_i$, we have $H_i(X; \mathbb{R}) \approx \mathcal{H}_i$ with the isomorphism induced by $\pi_{H_i} : Z_i \twoheadrightarrow \mathcal{H}_i$.

4.3.2 For cohomology one uses the cochain complex

$$C^* = C^*(X) = \hom_{\mathbb{R}}(C_*(X), \mathbb{R}),$$

with differential $\delta^{i-1} = \hom_{\mathbb{R}}(d_i, 1) : C^{i-1} \to C^i$. The inner product of C_* induces a natural isomorphism

$$\Lambda_i : C_i \to C^i X$$

where $\sigma \longmapsto \langle \sigma, \ \rangle$ for each i-cell σ of X. Since $(\Lambda_{i+1})(\sigma)(c) = \langle \delta_i \sigma, c \rangle = \langle \sigma, d_{i+1} c \rangle = (\Lambda_i \sigma)(d_{i+1} c) = (\delta^i \Lambda_i)(\sigma)(c)$, $\Lambda_* : (C_*, \delta_*) \to (C^*, \delta^*)$ defines an isomorphism of cochain complexes with $Z^i(X) \overset{\approx}{\to} \operatorname{Ker} \delta^i$, $B^i(X) \overset{\approx}{\to} \operatorname{Im} \delta^{i-1}$. This justifies the terminology Z^i = cocycles, B^i = coboundaries, that we have been using, even though Z^i, $B^i \subseteq C_i$.

4.3.3 Definition. For $f : X \to Y$, using the isomorphism in 4.3.1, we define $f_! : \mathcal{H}_iX \to \mathcal{H}_iY$ as the composition

$$\mathcal{H}_iX \overset{\approx}{\leftarrow} H_i(X;\mathbb{R}) \overset{f_*}{\to} H_i(Y;\mathbb{R}) \overset{\approx}{\to} \mathcal{H}_iY\,.$$

This makes $\mathcal{H}_i$ a covariant functor on the category of finite CW-complexes and continuous maps, also $f_!$ depends only on the homotopy class of f.

Similarly one can define $f^! : \mathcal{H}_iY \to \mathcal{H}_iX$ using

$$\mathcal{H}_iY \overset{\approx}{\to} H^i(Y;\mathbb{R}) \overset{f^*}{\to} H^i(X;\mathbb{R}) \overset{\approx}{\leftarrow} \mathcal{H}_iX\,.$$

This makes $\mathcal{H}_i$ into a contravariant functor, one easily checks that $f_!$, $f^!$ are adjoints.

CHAPTER V
INFINITE COMPLEXES AND ℓ_2-HOMOLOGY

The preliminary work in the previous chapters will now reap its dividends. With minor modifications of the definitions in Chapter IV, the ℓ_2-chains and ℓ_2-homology of an infinite (or finite) CW-complex Y will now be defined, taking into account a group action of some group G on Y, where the action is free, cocompact, and cellular, i.e. G permutes the cells of Y. The ℓ_2-chains (homology) will all be Hilbert G-modules.

5.1 Description of the ℓ_2-chains

5.1.1 Regular cellular coverings. The general situation considered from now on is that of a group G acting freely and cellularly on a connected CW-complex Y. In this case, denoting the orbit CW-complex by $X = Y/G$, the covering projection $p : Y \twoheadrightarrow X$ is a regular covering, i.e. $p_*\pi_1(Y, y_0)$ is a normal subgroup of $\pi_1(X, x_0)$. We also assume the action of G on Y to be cocompact, so that X is a compact (hence finite) CW-complex.

5.1.2 Example. A simple but useful example to keep in mind is the standard universal covering projection $p : \mathbb{R} \twoheadrightarrow S^1$, with $G = \mathbb{Z}$ acting on $\mathbb{R}$ by translations by integers. Similarly, the universal cover of any compact CW-complex can serve as an example.

5.1.3 Remarks. (a) Since X is finite, $\pi_1(X)$ is countable, hence so is $G = \pi_1(X)/p_*\pi_1(Y)$ (cf. the introduction to Chapter III).
(b) Also note that in this situation the ordinary (integral) chain group $K_i(Y)$ is a finitely generated free module over the group ring $\mathbb{Z}G$, with rank equal to the number of i-cells of X, and d_i is a $\mathbb{Z}G$ map.

5.1.4 Definition. $C_i^{(2)}(Y) := \{\sum_{\sigma\in J_i} f(\sigma)\sigma : f(\sigma) \in \mathbb{R}, \sum_{\sigma\in J_i}(f(\sigma))^2 < \infty\}$, the square summable chains of Y, where J_i is the set of i-cells of Y. Clearly $C_i^{(2)}(Y)$ is a Hilbert space with orthonormal basis J_i.

5.1.5 Definition. The ℓ_2-chains of Y are

$$C_i(Y, G) = l_2G \otimes_{\mathbb{Z}G} K_i(Y)\,,$$

which we often write simply $C_i(Y)$ if no confusion is possible. Note that since ℓ_2G is a (left) $\mathbb{R}G$-module, so is $C_i(Y)$ via $\mathbb{R}G\otimes_{\mathbb{R}}(\ell_2G\otimes_{\mathbb{Z}G}K_i(Y)) \approx (\mathbb{R}G\otimes_{\mathbb{R}}\ell_2G)\otimes_{\mathbb{Z}G}K_i(Y) \to \ell_2G \otimes_{\mathbb{Z}G} K_i(Y) = C_i(Y)$.

The group G permutes the set J_i of i-cells, choose from each orbit a representative $\bar{\tau}^i_\mu$, $\mu \in 1, \cdots, \alpha_i$, where α_i is the number of i-cells of X. The collection $\{x \otimes \bar{\tau}^i_\mu : x \in G, \mu \in \{1, ..., \alpha_i\}\}$ then may be taken as an orthonormal basis for $C_i(Y)$. The Hilbert space structure thus induced on $C_i(Y)$ is independent of the choice of representatives $\bar{\tau}^i_\mu$. Indeed, we have

5.1.6 Proposition. *$C_i(Y)$ and $C_i^{(2)}(Y)$ are naturally isomorphic as Hilbert spaces.*

Proof. The canonical bijection $\{x \otimes \bar{\tau}^i_\mu\} \to J_i$, given by $x \otimes \bar{\tau}^i_\mu \mapsto x \cdot \bar{\tau}^i_\mu$, defines a canonical bijection between the respective orthonormal Hilbert bases of $C_i(Y)$ and $C_i^{(2)}(Y)$, and hence extends to a natural isomorphism of Hilbert spaces. □

Note also that if $f \in \ell_2 G$ then $\|f \otimes \bar{\tau}^i_\mu\| = \left\| \sum f(x) x \otimes \bar{\tau}^i_\mu \right\| = \|f\|$, proving

5.1.7 Proposition. *The map $(\ell_2 G)^{\alpha_i} \to C_i Y$, $(f_1, \cdots, f_{\alpha_i}) \mapsto \sum_{\mu=1}^{\alpha_i} f_\mu \otimes \bar{\tau}^i_\mu$, defines an isometric G-equivariant isomorphism of Hilbert spaces.*

5.1.8 Definition. The boundary map on $C_*(Y)$ is

$$C_i(Y) = \ell_2 G \otimes_{\mathbb{Z}G} K_i(Y) \xrightarrow{\ell_2 G \otimes d_i} \ell_2 G \otimes_{\mathbb{Z}G} K_{i-1}(Y) = C_{i-1}(Y).$$

Taking the $\mathbb{Z}G$ bases $\{\bar{\tau}^i_\mu : \mu = 1, \cdots, \alpha_i\}$ and $\{\bar{\tau}^{i-1}_\nu : \nu = 1, \cdots, \alpha_{i-1}\}$ for $K_i(Y)$ and $K_{i-1}(Y)$ respectively, d_i (being G-equivariant) represents a morphism $(\mathbb{Z}G)^{\alpha_i} \to (\mathbb{Z}G)^{\alpha_{i-1}}$ of $\mathbb{Z}G$-modules, so also via the inclusion $\mathbb{Z} \subseteq \mathbb{R}$ a morphism $(\mathbb{R}G)^{\alpha_i} \to (\mathbb{R}G)^{\alpha_{i-1}}$ of $\mathbb{R}G$-modules.

5.1.9 Proposition. *The boundary map $\ell_2 G \otimes_{\mathbb{Z}G} d_i$ is a bounded operator.*

Proof. It is identical to the map

$$\ell_2 G \otimes_{\mathbb{R}G} d_i : \ell_2 G \otimes_{\mathbb{R}G} K_i(Y; \mathbb{R}) \to \ell_2 G \otimes_{\mathbb{R}G} K_{i-1}(Y; \mathbb{R});$$

now apply 3.1.9. □

We usually write simply d_i for $\ell_2 G \otimes_{\mathbb{R}G} d_i$.

5.2 Unreduced and reduced ℓ_2-homology

We have seen in § 5.1 that the operators $d_i : C_i(Y) \to C_{i-1}(Y)$ are bounded (hence continuous) and G-equivariant. The same then holds for their adjoints $\delta_{i-1} = d_i^*$ (cf. 2.3.7(a) and 3.1.10(a)). As in § 4.1 put $\operatorname{Ker} d_i = Z_i(Y) = Z_i$, $\operatorname{Ker} \delta_i = A^i$, and $\mathcal{H}_i = Z_i \cap Z^i$; these are all G-invariant Hilbert subspaces of $C_i = C_i(Y)$. Similarly define $B_i = \operatorname{Im} d_{i+1}$, $B^i = \operatorname{Im} \delta_{i-1}$; these are G-invariant linear subspaces of C_i but in general are not closed. Then, just as in §4.1, we have the next two results.

5.2.1 ℓ_2-Hodge-de Rham decomposition. $C_i = \bar{B}^i \boxplus Z_i = \bar{B}_i \boxplus Z^i = \bar{B}^i \boxplus \bar{B}_i \boxplus \mathcal{H}_i$.

5.2.2 Proposition. *$\mathcal{H}_i = \operatorname{Ker} \Delta_i$, where $\Delta_i = d_{i+1}\delta_i + \delta_{i-1} d_i$ is the ℓ_2-Laplacian (note $\mathcal{H}_i = \mathcal{H}_i(Y, G)$ is sometimes written for extra clarity).*

The proofs are identical to those in § 4.1, apart from a little extra care, using continuity, to first establish that the ℓ_2-analogue of Proposition 4.1.5, with B_i, B^i replaced respectively by $\bar{B}_i$, $\bar{B}^i$, is valid.

5.2.3 Definition. (a) $H_i(Y) = Z_i / B_i$, (b) ${}^{\text{red}}H_i(Y) = Z_i / \bar{B}_i$.

5.2.4 Caution. The reduced ℓ_2-homology groups ${}^{\mathrm{red}}H_i(Y)$ have nothing in common with the notion of reduced homology $\tilde{H}_i(Y)$ in usual homology theory.

5.2.5 Proposition. ${}^{\mathrm{red}}H_i(Y) \approx \mathcal{H}_i(Y, G)$, induced by $\pi_{\mathcal{H}_i} : Z_i \to \mathcal{H}_i$.

Proof. Same as 4.3.1, with B_i replaced by $\bar{B}_i$.

5.2.6. Similarly, define ${}^{\mathrm{red}}H^i(Y) = Z^i/\bar{B}^i$, and again one has ${}^{\mathrm{red}}H_i(Y) \approx \mathcal{H}_i(Y, G)$.

5.2.7 Remark. For G finite $B_i = \bar{B}_i$, $B^i = \bar{B}^i$ and everything reduces to the situation of 4.3, with $\mathcal{H}_i(Y, G) \approx H_i(Y; \mathbb{R}) \approx H^i(Y; \mathbb{R})$.

5.2.8 Remark. C_i, Z_i, Z^i, $\bar{B}_i$, $\bar{B}^i$, $\mathcal{H}_i$, and ${}^{\mathrm{red}}H_i(Y)$, ${}^{\mathrm{red}}H^i(Y)$ are all clearly Hilbert G-modules, C_i being isomorphic to $(\ell_2 G)^{\alpha_i}$ and the others being G-invariant submodules of C_i or quotients of G-invariant submodules.

The non-reduced (co-)homology groups are generally less useful and more difficult to compute. They are not in general Hilbert G-modules.

5.2.9 Definition. (a) For the non-reduced ℓ_2-homology,

$$Z_i/B_i \approx H_i(\ell_2 G \otimes_{\mathbb{Z}G} K_*(Y)) := H_i^G(Y; \ell_2 G),$$

the equivariant homology of Y with coefficients in the G-module $\ell_2 G$.
(b) $H_G^i(Y; \ell_2 G) := H^i(C^*(Y)) = H^i(\mathrm{hom}_{\mathbb{Z}G}(K_*(Y), \ell_2 G)$ is the equivariant cohomology of Y with coefficients in $\ell_2 G$.

For equivariant homology, the inclusion $B_i \hookrightarrow \bar{B}_i$ induces a natural surjection $H_i^G(Y; \ell_2 G) \twoheadrightarrow {}^{\mathrm{red}}H_i(Y)$.

5.2.10 Remark. For some applications, it is useful to note that everything done in this section generalizes to the case of a regular covering $Y \twoheadrightarrow X$ where the k-skeleton $X^{(k)}$ of X is finite, provided we consider $C_i(Y)$ only for $i < k$.

5.2.11 Definition. The "canonical" map can_i: $H_i(Y; \mathbb{R}) \to {}^{\mathrm{red}}H_i(Y)$ is defined by simply considering any ordinary real cycle as an ℓ_2-cycle (with finite support).

5.2.12 Definition. The "canonical" map can^i: ${}^{\mathrm{red}}H^i(Y) \to H^i(Y; \mathbb{R})$ is defined as follows. Consider $x = [\xi] \in {}^{\mathrm{red}}H^i(Y) = Z^i/\bar{B}^i$, since $Z^i = \bar{B}^i \boxplus \mathcal{H}_i$, $\xi = \gamma + \eta$ uniquely with $\gamma \in \bar{B}^i$, $\eta \in \mathcal{H}_i$. Then $\xi - \eta = \gamma \in \bar{B}^i \Rightarrow x = [\xi] = [\eta]$, i.e. x has a unique harmonic cocycle representative $\eta \in \mathcal{H}_i$. As in 4.3.2, η identifies under the isomorphism Λ_* with an ordinary cocycle in $C^*(Y; \mathbb{R})$, which thus defines a cohomology class $can^i(x) \in H^i(Y; \mathbb{R})$.

CHAPTER VI
PROPERTIES OF ℓ_2-HOMOLOGY

6.1 G-HOMOTOPY INVARIANCE

Probably the most important property of $\mathcal{H}_i(Y)$ is showing that, up to isomorphism (as a Hilbert G-module), it depends only on the G-homotopy type of Y. By 5.2.5, this is equivalent to showing the same for ${}^{\mathrm{red}}H_i(Y)$.

6.1.1 Lemma. *${}^{\mathrm{red}}H_i$ is a functor from the category of free cocompact G-CW complexes and G-homotopy classes of maps, to the category $\mathfrak{H}G$.*

Proof. Let $f : Y \to Z$ be a G-map of free cocompact G-CW-complexes. By G-cellular approximation $f \simeq_G g$, a cellular G-map. By 3.1.9, $g_* = g_i : C_i(Y) \to C_i(Z)$ are bounded operators, and the g_i also as usual are chain maps. By continuity $g_i(\bar{B}_i(Y)) \subseteq \bar{B}_i(Z)$ so it induces ${}^{\text{red}}H_i(g) : {}^{\text{red}}H_i(Y) \to {}^{\text{red}}H_i(Z)$. If $h : Y \to Z$ is a cellular G-map G-homotopic to g, then $K_i g, K_i h : K_i Y \to K_i Z$ are chain homotopic morphisms of G-chain complexes. Hence $\ell_2 G \otimes_{\mathbb{R}G} K_* g := g_*$, $\ell_2 G \otimes_{\mathbb{R}G} K_* h := h_*$ are chain homotopic as well. Thus $(g_* - h_*)(Z_i(Y)) \subseteq B_i(Y) \subseteq \bar{B}_i(Y)$, whence ${}^{\text{red}}H_i(g_*) = {}^{\text{red}}H_i(h_*)$ for all i, showing that ${}^{\text{red}}H_i(g_*)$ depends only on the G-homotopy class of f. □

6.1.2 Corollary. *The Hilbert G-modules $\mathcal{H}_i(Y)$ are also functorial and give rise to G-homotopy invariants.*

6.1.3 Corollary. *If $f : Y \to Z$ is a G-map between free cocompact G-CW-complexes and is also a homotopy equivalence, then $\mathcal{H}_i(Y) \approx \mathcal{H}_i(Z)$.*

Proof. Indeed f induces a weak equivalence ${}^{\text{red}}H_i(Y) \to {}^{\text{red}}H_i(Z)$, thus ${}^{\text{red}}H_i(Y) \approx {}^{\text{red}}H_i(Z)$ and $\mathcal{H}_i(Y) \approx \mathcal{H}_i(Z)$, all as Hilbert G-modules.
Remark. There is no need, in this last corollary, to assume that f is a G-homotopy equivalence; it is well known that any G-map between free G-CW-complexes which is a homotopy equivalence is also a G-homotopy equivalence.

The situation for ${}^{\text{red}}H^i(Y) = Z^i(Y)/\bar{B}^i(Y)$ is similar, giving contravariant functors with ${}^{\text{red}}H^i(f)$ being induced by the adjoint $f_i^* : C_i(Z) \to C_i(Y)$. One also has ${}^{\text{red}}H_i(Y) \approx \mathcal{H}_i(Y) \approx {}^{\text{red}}H^i(Y)$ as well.
The next simple lemma will be useful in the examples to follow.
6.1.4 Lemma. *If G is infinite then for $n \geq 1$ the left G-module $(\ell_2 G)^n$ contains no G-invariant element besides 0.*

Proof. If $f = \sum_{x \in G} f(x)x \in \ell_2 G$ is G-invariant, then for each $y \in G$ $f = y \cdot f = \sum f(y^{-1}x)x \Rightarrow f(x) = f(y^{-1}x)$ for all $y \in G$ which implies that $f(x)$ is constant. Since $\sum (f(x))^2$ converges and G is infinite, $f(x) = 0$ for all $x \in G$, i.e. $f = 0$. Similarly for $n > 1$. □

6.1.5 Example. Suppose Y is a connected G-CW complex with cocompact 1-skeleton and $|G| = \infty$. Then ${}^{red}H_0(Y) = \mathcal{H}_0(Y) = {}^{red}H^0(Y) = 0$. To see this, first note that $K_1(Y) \xrightarrow{d_1} K_0(Y) \xrightarrow{\epsilon} \mathbb{Z} \to 0$ is exact, and since $\otimes$ is a right exact functor

$$C_1(Y) \xrightarrow{d_1} C_0(Y) \twoheadrightarrow \ell_2 G \otimes_G \mathbb{Z} \to 0$$

is also exact. Hence $(\operatorname{Im} d_1)^\perp = \operatorname{Ker} \delta^0 \subseteq C_0(Y)$ is mapped injectively (indeed isomorphically) to $\ell_2 G \otimes_G \mathbb{Z}$, whence $\operatorname{Ker} \delta^0$ consists of G-invariant elements and is 0 by Lemma 6.1.4.

For any discrete group G, its classifying space BG is an Eilenberg-MacLane space $K(G,1)$, and its universal cover is written EG. The projection $EG \to K(G,1)$ is an example of a regular covering with free cellular action of G on EG, but in general $K(G,1)$ need not be compact. However, if G is finitely generated, then we can (and do) take a cellular decomposition for $K(G,1)$ with a single 0-cell, and a single 1-cell for each generator of G, thus the 1-skeleton is a finite wedge of circles.

In particular, for any finitely generated infinite group G, Example 6.1.5 applies and $\mathcal{H}_0(EG) = 0$. On the other hand, if G is finite, the proof of 6.1.4 shows that the G-invariant elements of $\ell_2 G$ are the constants, i.e. $\mathbb{R}$, and thus $\mathcal{H}_0(EG) \approx \mathbb{R}$.

In the next example we see that in general, in the same situation, the unreduced $H_0(Y)$ is not 0.

6.1.6 Example. Consider now the same example in the Introduction, $Y = \mathbb{R} \twoheadrightarrow X = S^1$, $G = \mathbb{Z}$, $S^1 = e^0 \cup e^1$. In this case the ℓ_2-chain complex $C_*(Y)$ is

$$0 \to \ell_2(\mathbb{Z}) \xrightarrow{d_1} \ell_2(\mathbb{Z}) \to 0\,.$$

Let x generate $\mathbb{Z} = \{x^n\}$, write $f \in \ell_2(\mathbb{Z})$ as $f = \sum_{n\in\mathbb{Z}} a_n x^n$, $a_n \in \mathbb{R}$. Since $d_1(\bar{e}^1_n) = \bar{e}^0_{n+1} - \bar{e}^0_n = (x-1)\bar{e}^0_n$, $d_1(f) = (x-1)f$. Then clearly d_1 is injective (same argument as in 6.1.4), and from Example 6.1.5 ${}^{red}H_0(Y) = 0$ implies that $\operatorname{Im} d_1$ is dense in $\ell_2(\mathbb{Z})$. But d_1 is not surjective, e.g. $1 \notin \operatorname{Im} d_1$. To see this, if $1 \in \operatorname{Im} d_1$ then $1 = (x-1)\sum a_i x^i$ implies that all a_j, $j < 0$, are equal and hence 0, whence $a_{-1} - a_0 = 1$ implies that $a_0 = -1 = a_1 = a_2 = \ldots$, which is impossible. Therefore $H_0^G(Y; \ell_2 G) \neq 0$, in contrast to ${}^{red}H_0(Y) = 0$.

Equivalently, this example shows that $\mathcal{H}_i(E\mathbb{Z}) = 0$, $i \geq 0$, whereas $H_i^{\mathbb{Z}}(E\mathbb{Z}; \ell_2\mathbb{Z}) = H_i(\mathbb{Z}; \ell_2\mathbb{Z}) = 0$ for $i > 0$ but $H_0(\mathbb{Z}; \ell_2\mathbb{Z}) \neq 0$. For cohomology, $H^1(\mathbb{Z}; \ell_2\mathbb{Z}) \neq 0$ and otherwise $H^i(\mathbb{Z}; \ell_2\mathbb{Z}) = 0$, $i \neq 1$. For a more systematic study of the difference between reduced and unreduced ℓ_2-homology, the reader is referred to the initial work of Novikov and Shubin [10] and its further developments by Farber [5] and Lück [8].

6.2 $\ell_2 G$-CHAIN COMPLEXES

6.2.1 Definition. A chain complex

$$V_* : \ldots \to V_{i+1} \xrightarrow{d_{i+1}} V_i \xrightarrow{d_i} V_{i-1} \to \ldots$$

of Hilbert G-modules V_i is called an ℓ_2-chain complex if each d_i is a bounded G-equivariant operator. Of course $d_i d_{i+1} = 0$ for all i also holds (the chain complex condition).

6.2.2 Definitions. With V_* as above,
(a) ${}^{\text{red}}H_i(V_*) = \operatorname{Ker} d_i / \overline{\operatorname{Im} d_{i+1}}$,
(b) V_* is called weak exact if ${}^{\text{red}}H_i(V_*) = 0\ \forall\ i$.

6.2.3 Definitions. Let V_*, W_* be ℓ_2-chain complexes.
(a) A morphism $\phi_* : V_* \to W_*$ is an ordinary chain map with each ϕ_i a bounded G-equivariant operator,
(b) two morphisms ϕ_*, ψ_* are $\ell_2 G$-homotopic if they are chain homotopic by a chain homotopy of of bounded G-equivariant operators,
(c) two $\ell_2 G$-chain complexes are $\ell_2 G$-chain equivalent if there exist $\phi_* : V_* \to W_*$ and $\psi_* : W_* \to V_*$, both morphisms as in (a), with $\psi_*\phi_* \simeq id_{V_*}$, $\phi_*\psi_* \simeq id_{W_*}$ in the sense of (b).

Just as in the ordinary homology theory, a morphism $\phi_* : V_* \to W_*$ induces well defined bounded G-equivariant operators ${}^{\text{red}}H_i(V_*) \to {}^{\text{red}}H_i(W_*)$ depending only on the $\ell_2 G$-homotopy class of ϕ_*. The proof is identical to the usual one apart from the extra use of continuity to show that $\phi_i(\overline{\operatorname{Im} d_{i+1}}) \subseteq \overline{\operatorname{Im} d'_{i+1}}$, where d, d' are the respective boundary operators.

6.2.4 Corollary. *If V_*, W_* are ℓ_2-chain equivalent ℓ_2-chain complexes, then* ${}^{\mathrm{red}}H_i(V_*) \approx {}^{\mathrm{red}}H_i(W_*)$ *as Hilbert G-modules, for all i.*

6.2.5 Definition. A weak exact ℓ_2G-chain complex $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ is called a short weak-exact sequence (of Hilbert G-modules). Then α is injective, $\overline{\operatorname{Im}\beta} = W$, and $\operatorname{Ker}\beta = \overline{\operatorname{Im}\alpha}$.

6.2.6 Proposition. *For a short weak-exact sequence of Hilbert G-modules $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$, $\dim_G V = \dim_G U + \dim_G W$.*

Proof. From 3.2.6, $\overline{\beta(V)} \approx (\operatorname{Ker}\beta)^{\perp} \approx V/\operatorname{Ker}\beta$, hence

$$\dim_G V = \dim_G(\operatorname{Ker}\beta) + \dim_G(\operatorname{Ker}\beta)^{\perp} = \dim_G(\overline{\operatorname{Im}\alpha}) + \dim_G(\overline{\operatorname{Im}\beta}),$$

i.e. $\dim_G(V) = \dim_G U + \dim_G W$, since α is injective and $\overline{\operatorname{Im}\beta} = W$. □

6.2.7 Corollary. *Let $V_* : 0 \to V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \dots V_0 \xrightarrow{d_0} 0$ be a chain complex of Hilbert G-modules. Then*

$$\sum_{i\geq 0}(-1)^i \dim_G V_i = \sum_{i\geq 0} \dim_G {}^{\mathrm{red}}H_i(V_*)\,.$$

Proof. Set $K_i = \operatorname{Ker} d_i$, $I_i = \overline{\operatorname{Im} d_{i+1}}$. Then we have short weak-exact sequences of Hilbert G-modules

$$0 \to K_i \hookrightarrow V_i \to I_{i-1} \to 0\,,$$
$$0 \to I_i \hookrightarrow K_i \to {}^{\mathrm{red}}H_i(V_*) \to 0\,.$$

Hence, $\dim_G V_i = \dim_G K_i + \dim_G I_{i-1}$, and $\dim_G {}^{\mathrm{red}}H_i = \dim_G K_i - \dim_G I_i$, and the result follows easily (as for ordinary chain complexes). □

6.3 ℓ_2-Betti numbers

6.3.1 Definition. Let Y be a free cocompact G-CW complex. The i-th ℓ_2-Betti number of Y (with respect to G) is

$$\beta_i(Y;G) := \dim_G {}^{\mathrm{red}}H_i(Y)\,.$$

Applying results from § 3.3 and § 6.1, 6.2 gives immediately

6.3.2 Proposition.

(a) *$\beta_i(Y,G)$ is a G-homotopy invariant of Y, and therefore a homotopy invariant of $X = Y/G$,*

(b) *If H is a subgroup of G with $[G:H] = m < \infty$, then $\beta_i(Y;H) = m\beta_i(Y;G)$ (here $Y/H \twoheadrightarrow Y/G$ is an m-sheeted covering projection),*

(c) *If $|G| < \infty$, then $\beta_i(Y;G) = \frac{1}{|G|}b_i(Y)$, in particular if Y is connected then $\beta_0(Y;G) = \frac{1}{|G|}$,*

(d) *If $|G| = \infty$ and Y is connected then $\beta_0(Y;G) = 0$ (cf. 6.1.5).*

6.3.3 Definition. Let X be a connected finite CW-complex. Then we define the ℓ_2-Betti number of X by $\beta_i(X) = \beta_i(\tilde{X};G)$, where $\tilde{X}$ is the universal covering space of X and $G = \pi_1(X,x_0)$.

Here are two very simple properties.

6.3.4 Proposition. $0 \le \beta_i(X) \le \alpha_i$.

Proof. Clear since ${}^{\text{red}}H_i(\tilde{X})$ is a subquotient of $(\ell_2 G)^{\alpha_i}$, as Hilbert G-modules. □

6.3.5 Proposition. *If $\hat{X}$ is a connected m-sheeted covering of X with $H = \pi_1(\hat{X})$, $G = \pi_1(X)$, then $\beta_i(\hat{X}) = m\beta_i(X)$.*

Proof. $\beta_i(\hat{X}) = \dim_H {}^{\text{red}}H_i(\hat{X}) = m \cdot \dim_G {}^{\text{red}}H_i(\hat{X}) = m \cdot \beta_i(X)$. □

6.3.6 Example. Note that this is very different from the behaviour of ordinary Betti numbers, e.g. $S^1 \xrightarrow{s} S^1$, $s(z) = z^2$, is a 2-sheeted covering but $b_1(S^1) = 1 = b_1(\hat{X})$ and $b_1(X) = b_1(S^1) = 1$. From 6.1.6, $\beta_i(S^1) = 0$, for all $i \ge 0$.

6.3.7 Example. More generally, if X (like S^1) is a connected finite CW-complex which possesses a regular finite covering space $\hat{X} \to X$ of degree $m > 1$, with $\hat{X}$ homotopy equivalent to X, then $\beta_i(X) = 0$ for all i, since $\beta_i(X) = \beta_i(\hat{X}) = m \cdot \beta_i(X)$.

6.3.8 Atiyah's conjecture. In [1] Atiyah asked whether the ℓ_2-Betti numbers are rational, or even integral in the case of a torsion free group G.

A somewhat weaker conjecture is the

Zero-divisor conjecture. For G torsion free, $\mathbb{Q}[G]$ contains no non-trivial zero divisors.

It is not very difficult to show the first conjecture implies the second. The zero-divisor conjecture has been proved for a large class of torsion free groups, cf. [7].

We close this section with theorems on the Euler characteristic and Morse inequalities involving ℓ_2-Betti numbers. The proofs are analogous to those in § 4.2, using the machinery in § 6.2, and are omitted.

6.3.9 Theorem. *For X a finite connected CW-complex,*

$$\chi(X) = \sum_{i \ge 0} (-1)^i \beta_i(X).$$

6.3.10 Theorem. *For X as in 6.3.8, let N be a normal subgroup of $\pi_1(X)$ with $\pi_1(X)/N = Q$. Let $\hat{X}$ denote the covering space associated with N. Then*

$$\chi(X) = \sum_{i \ge 0} (-1)^i \beta_i(\hat{X}; Q).$$

6.3.11 Morse inequalities. Let X be a connected CW-complex with finite $(k+1)$-skeleton. Then

$$\alpha_k - \alpha_{k-1} + \ldots + (-1)^k \alpha_0 \ge \beta_k - \beta_{k-1} + \ldots + (-1)^k \beta_0.$$

CHAPTER VII
APPLICATIONS

At the end of Chapter VI, applications have already been made (of the ℓ_2-Betti numbers) to Euler characteristic and Morse inequalities. Further applications and a few simple computations will be sketched in the present chapter. Proofs and many details will often be omitted here, but it is hoped that the reader will at least get a taste of some of the significant applications that are possible.

7.1 Two examples

7.1.1 Example. This example extends Example 6.1.6, $X = S^1$, to the wedge of k circles $X = \bigvee_{i=1}^{k} S^1$. Here $\pi_1(X) = G = F_k$, the free group on k generators. The universal cover $Y = \tilde{X}$ is a tree (as is the universal cover of any connected graph) and thus contractible, hence $X = K(G,1)$ is an Eilenberg-MacLane space. Furthermore, the cellular structure $e^0, e^1_1, \ldots, e^1_k$ shows that $\chi(X) = 1 - k$. By 6.1.5, and Definition 6.3.3, $\beta_0(X) = \beta_0(Y;G) = 0$. Hence, since $\beta_i(X) = 0$ trivially for $i > 1$, we have $\chi(X) = 1 - k = 0 - \beta_1(X)$, which implies that $\beta_1(X) = k - 1$, and $\beta_i(X) = 0$ otherwise.

7.1.2 Example. Let $\sum_g = X$ be an orientable surface of genus $g > 0$, with $G = \sigma_g := \pi_1(\sum_g)$ (e.g. $\sum_1 = S^1 \times S^1, \sigma_1 \approx \mathbb{Z} \times \mathbb{Z}$, and for $g > 1$ σ_g is an infinite non-abelian group). Here $\chi(\sum_g) = 2 - 2g$ is well known, and $Y = X = \mathbb{R}^2$. Again, $\sum_g = K(G,1)$ is an Eilenberg-MacLane space, and similar to the previous example $\beta_0(X) = 0$. In this case, since X also happens to be a 2-manifold, Poincaré duality (an ℓ_2 version, not hard to prove but omitted here) shows that $\beta_2(X) = \beta_0(X) = 0$. Then $2 - 2g = \chi(X) = 0 - \beta_1(X) + 0$ implies that $\beta_1(X) = 2g - 2$, $\beta_i(X) = 0$ otherwise.

These two examples have an algebraic interpretation.

7.1.3 Definition. Let G be a (necessarily finitely presented) group with a finite CW-model $K(G,1)$, then

$$\beta_1(G) := \beta_1(K(G,1)).$$

From Remark 5.2.10 we can extend this definition as follows.

7.1.4 Definition. Let G be a group with a CW-model $K(G,1)$ having finite n-skeleton for some $n \geq 2$ (called a group of type F_n), then

$$\beta_i(G) := \beta_i(K(G,1)^{(n)}), \qquad i < n.$$

In particular β_1 is defined for any finitely presented group.

7.2 Deficiency of groups

The deficiency of a finitely presented group G was defined in the Introduction 1.1.3. We now give the proof of the first easy result mentioned there.

7.2.1 Proposition. $\operatorname{def}(G) \leq b_1(G)$.

Proof. By definition $b_1(G) = \operatorname{rank}(H_1(K(G,1)))$, and

$$H_1(K(G,1)) = \pi_1(K(G,1))_{ab} = G_{ab},$$

since by definition $\pi_1(K(G,1)) = G$. Hence

$$b_1(G) = \operatorname{rank}(G_{ab}) = \dim_{\mathbb{R}}(G_{ab} \otimes \mathbb{R}) \geq g - r,$$

since $G_{ab} \otimes \mathbb{R}$ is given as real vector space with g generators and r (linear homogeneous) relations. □

Our second result involves ordinary Betti numbers and refines the first proposition, then we turn to a few sample results with ℓ_2-Betti numbers.

7.2.2 Proposition. $\operatorname{def}(G) \leq b_1(G) - b_2(G)$.

Proof. For G given with g generators and r relations, we can construct $K(G,1)$ with $K(G,1)^{(2)}$ having one 0-cell, g 1-cells, r 2-cells. Then, using 6.3.9,

$$\chi(K(G,1)^{(2)}) = r - g + 1 = 1 - b_1(K(G,1)^{(2)}) + b_2(K(G,1)^{(2)}).$$

But $b_1(K(G,1)^{(2)}) = b_1(G)$ and $b_2(K(G,1)^{(2)}) \geq b_2(G)$, thus $r - g \geq b_2(G) - b_1(G)$, or $g - r \leq b_1(G) - b_2(G)$, as required. □

7.2.3 Theorem. $\operatorname{def}(G) \leq 1 + \beta_1(G)$.

Proof. Similar to 7.2.2, here we find that

$$\chi = r - g + 1 = \beta_0(G) - \beta_1(G) + \beta_2(K(G,1)^{(2)})$$

implies that

$$g - r = 1 + \beta_1(G) - \beta_0(G) - \beta_2(K(G,1)^{(2)}) \leq 1 + \beta_1(G). \qquad \square$$

7.2.4 Corollary. *If* $\beta_1(G) = 0$ *then* $\operatorname{def}(G) \leq 1$.

Suppose $K(G,1)^{(3)}$ is finite, i.e. G is of type F_3. The Morse inequalities 6.3.11 yield (with $k = 2$) $r - g + 1 \geq \beta_2(G) - \beta_1(G) + \beta_0(G)$, $g - r \leq 1 + \beta_1(G) - \beta_2(G)$. This proves

7.2.5 Theorem. *For any group G of type F_3,* $\operatorname{def}(G) \leq 1 + \beta_1(G) - \beta_2(G)$.

Note that the fundamental group of any closed (compact with empty boundary) 3-manifold will be of type F_3.

7.2.6 Proposition. *Let G be a free group of rank k, then* $\operatorname{def}(G) = k$.

Proof. Since G has a presentation with k generators and 0 relations, $\operatorname{def}(G) \geq k - 0 = k$. But by 7.1.1 and 7.2.3, $\operatorname{def}(G) \leq 1 + \beta_1(G) = 1 + (k-1) = k$. □

Similarly, using the standard presentation of σ_g with $2g$ generators $x_1, y_1, x_2, y_2, \ldots, x_g, y_g$, and the single relation $[x_1, y_1][x_2, y_2] \cdots [x_g, y_g] = e$, and using 7.1.2, 7.2.3, we have

7.2.7 Proposition. $\operatorname{def}(\sigma_g) = 2g - 1$.

7.3 Amenable groups

Let G be a group and $B = \{f : G \xrightarrow{f} \mathbb{R},\ f \text{ bounded}\}$. Consider B as a G-module by putting $(x \cdot f)(y) = f(yx)$ for all $x, y \in G$ and $f \in B$.

7.3.1 Definition. A mean on G is a linear map $M : B \to \mathbb{R}$ such that for all $x \in G$ and $f \in B$,
(a) $M(1) = 1$ ($1 =$ the constant function 1),
(b) $M(x \cdot f) = M(f)$,
(c) $f \geq 0 \Rightarrow M(f) \geq 0$.

7.3.2 Definition. A group G is amenable if it admits a mean.

A finite group G is amenable, indeed M is uniquely given by

$$M(f) = \frac{1}{|G|} \sum_{x \in G} f(x) .$$

The question of determining all infinite amenable groups is deep and has led to much interesting work. In particular every abelian and indeed every solvable group is amenable. Here, without proof, is a useful lemma of the theory (cf. [3], [4]).

7.3.3 Cheeger-Gromov Lemma. *Let Y be a connected free cocompact G-CW complex and G an infinite amenable group. Then*

$$\mathrm{can}^i : {}^{\mathrm{red}}H_i(Y) \to H^i(Y;\mathbb{R}) ,$$

as defined in 5.2.12, is injective for all $i \geq 0$.

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