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EQUIVARIANT DIFFERENTIAL OPERATORS

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1. EQUIVARIANT DIFFERENTIAL OPERATORS ON HOMOGENEOUS SPACES

1.1. Homogeneous vector bundles. We consider homogeneous spaces of the form $S = G/H$ where G is a semisimple Lie group with finite center and H a closed subgroup.

If (ρ, V) is a representation of H, then the homogeneous vector bundle $G \times_H V$ associated with this representation is the quotient space of $G \times V$ under the equivalence relation

$$(g, v) \sim (gh, \rho(h^{-1})v).$$

The homogeneous vector bundle $G \times_H V$ is a bundle over G/H. The projection mapping

$$p : G \times_H V \longrightarrow G/H$$

maps the equivalence class of (g, h) to gH.

Example. *The tangent space TS of $S = G/H$.*

The Lie algebras of G and its subgroup H are denoted by g and h respectively. The tangent space of S at the identity coset $0 = eH$ is identified with g/h.

To an element $X + \mathrm{h} \in \mathrm{g}/\mathrm{h}$ there corresponds the equivalence class of curves through $0 \in S$ with representative $\exp tX\ H,\ t \in \mathbf{R}$. At the point $gH \in S$, the representative is

$$g \exp tX\ H = gh \exp t\mathrm{Ad}(h^{-1})X\ H.$$

An element in $T_{gH}(G/H)$ is therefore given by an equivalence class:

$$(g, X + \mathrm{h}) \sim (gh, \mathrm{Ad}(h^{-1})X + \mathrm{h}).$$

The tangent space $T(G/H)$ is thus the homogeneous vector bundle $G \times_H \mathrm{g}/\mathrm{h}$ and the representation of H on g/h is the adjoint representation on the quotient space

$$\mathrm{Ad}h(X + \mathrm{h}) = \mathrm{Ad}(h)X + \mathrm{h}.$$

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Fix a subspace $\mathfrak{q}$ of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$ and denote the projection onto $\mathfrak{q}$ along $\mathfrak{h}$ by $p_{\mathfrak{q}}$. Then $p_{\mathfrak{q}} \cdot \mathrm{Ad}$ is a representation of H on $\mathfrak{q}$.

Special case: *Symmetric spaces.*
If G is a semisimple Lie group of the non compact type and K a maximal compact subgroup, then one has the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

of the Lie algebra $\mathfrak{g}$ of G into an orthogonal (under the Killing form) direct product of the ± 1 eigenspaces of the Cartan involution ϑ. The subalgebra $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is a complementary vector subspace.

In this situation $\mathrm{Ad}(k)$ maps $\mathfrak{p}$ into itself (for all $k \in K$) and therefore

$$T(G/K) = G \times_K \mathfrak{p}$$

with K acting by Ad on $\mathfrak{p}$.

Given local trivialisations $U_0 \times V$ around $0 = eH$ and $U_g \times V$ around $g\,0 = g\,H$ left translation by g induces a mapping (denoted by τ_g) from a neighbourhood of $\{0\} \times V$ in $U_0 \times V$ to $U_g \times V$.
If $s : G/H \to G \times_H V$ is a section, then its lift $f_s : G \to V$ is defined by

$$f_s(g) = \tau_g^{-1} s(g\,0)\,.$$

Replacing g by gh, $h \in H$, this leads to

$$\begin{aligned} f_s(gh) &= (\tau_{gh})^{-1} s(gh\,0) = \tau_h^{-1} \tau_g^{-1} s(g\,0) \\ &= \tau_h^{-1} f_s(g) \end{aligned}$$

with τ_h a representation (ρ, V) of H on the vector space V.

The lifts of the C^∞−sections make up the space $C^\infty(G \times_H V)$ of C^∞−functions $f : G \to V$ satisfying

$$f(gh) = \rho(h^{-1}) f(g) \qquad g \in G,\ h \in H\,.$$

Conversely, to any $f \in C^\infty(G \times_h V)$ there corresponds a section s_f

$$s_f(g\,0) = (g, f(g))\,.$$

Example. *Vector fields on real hyperbolic space.*

$$\begin{aligned} SO(n,1) &= \{g \in SL(n+1) : \langle gx, gx\rangle = \langle x, x\rangle \quad \forall x \in \mathbb{R}^{n+1}\} \\ &\quad \text{with } \langle x, x\rangle = \textstyle\sum_{j=1}^n x_j^2 - x_{n+1}^2\,, \end{aligned}$$

$G = SO_0(n,1)$ (component in $SO(n,1)$ which contains the neutral element),
$K = SO(n)$.

The corresponding Lie algebra $\mathfrak{g}$ of $SO(n,1)$ decomposes as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with

$$\mathbf{k} = \left\{ \left(\begin{array}{c|c} \star & \\ \hline & 0 \end{array} \right) \text{ anti-symmetric} \right\} \qquad \text{Lie algebra of } K$$

$$\mathbf{p} = \left\{ \left(\begin{array}{c|c} & \star \\ \hline \star & \end{array} \right) \text{ symmetric} \right\} \qquad \text{complementary subspace.}$$

If $\vartheta(X) = -X^{tr}$ denotes the Cartan involution on $\mathbf{g}$, then $\mathbf{k}$ and $\mathbf{p}$ are its $+1$ and -1 eigenspaces respectively.

Real hyperbolic space is the symmetric space G/K. It can be realized as the unit ball

$$B = \{x \in \mathbb{R}^n : |x| < 1\}$$

with the hyperbolic metric $ds = \frac{dx}{(1-|x|^2)^{n-1}}$. The group G acts on B as the group of conformal transformations. (This is the Poincaré model of hyperbolic space.)

The natural action of the conformal transformations τ_g ($g \in G$) on the tangent space is given by

$$(\tau_g v)(x) = (\tau_g)_*(g^{-1}x) \;\; v(g^{-1}x)\,.$$

Set $f(g) = (\tau_g^{-1} v)(0)$. Then

$$\begin{aligned} f(gk) &= (\tau_{gk})_*^{-1} v(gk\,0) = \tau_{k*}^{-1}\tau_{g*}^{-1} v(g\,0) \\ &= \tau_{k*}^{-1} f(g)\,. \end{aligned}$$

In this case $\tau_{k*}^{-1}(0)$ is simply k^{-1}, if the standard representation of $K = SO(n)$ by its $n \times n$ matrix is written as k (in the following this representation will be denoted by δ_1).

In this particularly simple case, the representation Ad of K on p is just δ_1. This representation is self contragradient.

1.2. The gradient–construction of Stein-Weiss. If $S = G/K$ is a symmetric space as in the example above, with

$$\mathbf{g} = \mathbf{k} \dot{+} \mathbf{p}$$

and K acting on $\mathbf{p}$ by Ad, then for any $f \in C^\infty(G \times_K V)$ the gradient ∇f can be defined as an element in $C^\infty(G \times_K \operatorname{Hom}(\mathbf{p}, V))$:

$$\nabla f(g)(X) := Xf(g) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp tX)\,.$$

If g is replaced by gk

$$\begin{aligned}
\nabla f(gk)X &= \left.\frac{d}{dt}\right|_{t=0} f(gk\exp tX) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(g\exp \mathrm{Ad}(k)Xk) \\
&= \rho(k^{-1})\left.\frac{d}{dt}\right|_{t=0} f(\exp \mathrm{Ad}(k)X) \\
&= \rho(k^{-1})(\mathrm{Ad}(k)Xf)(g)\,.
\end{aligned}$$

The tensor representation $\mathrm{Ad}^\vee \otimes \rho$ on $\mathbf{p}^* \otimes V = \mathrm{Hom}(\mathbf{p}, V)$ is defined by

$$(\mathrm{Ad}^\vee \otimes \rho)(k^{-1})T = \rho(k^{-1}) \circ T \circ \mathrm{Ad}(k) \qquad T \in \mathrm{Hom}(\mathbf{p}, V)$$

thus ∇f is in $C^\infty(G \times_K \mathbf{p}^* \otimes V)$ with K acting by $\mathrm{Ad}^\vee \otimes \rho$ on $\mathbf{p}^* \otimes V$.

$$\nabla f(gk) = (\mathrm{Ad}^\vee \otimes \rho)(k^{-1})\nabla f(g)\,.$$

$\mathrm{Ad}^\vee$ denotes the contragradient representation: $\mathrm{Ad}^\vee(k) = \mathrm{Ad}^*(k^{-1})$. In general, the representation $\mathrm{Ad}^\vee \otimes \rho$ of K is not irreducible

$$\mathrm{Ad}^\vee \otimes \rho = \oplus_\ell\, m_\ell\, \rho_\ell, \qquad \mathbf{p}^* \otimes V = \oplus V_\ell^{m_\ell}$$

with multiplicities $m_\ell \in \mathbb{N}$. If pr_ℓ denotes the projection onto the space $V_\ell^{m_\ell}$, then $pr_\ell \circ \nabla$ is a G—equivariant differential operator from $C^\infty(G \times_K V)$ to $C^\infty(G \times_K V_\ell)$. This gradient construction is due to Stein and Weiss [SW].

Example. *Real hyperbolic space.*

Here, $\mathrm{Ad}^\vee$ on p is equivalent to the standard representation δ_1 of $K = SO(n)$. The projection operators on $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) = \mathbf{p}^* \oplus \mathbf{p}$ are determined by

$$\begin{aligned}
ST &= \frac{1}{2}(T + T^{tr}) - \frac{1}{n}(\mathrm{trace}\, T)I \\
AT &= \frac{1}{2}(T - T^{tr}) \\
DT &= \frac{1}{n}(\mathrm{trace}\, T)I
\end{aligned}$$

such that $T = ST + AT + DT$, for all $T \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$.

In the case of a symmetric space, the gradient operator ∇ is an equivariant operator, and so are the operators $pr_\ell \circ \nabla$. In the more general situation, when G/H is a homogeneous space, it can at best be hoped that $pr_\ell \circ \nabla$ is an equivariant operator.

Example. *The boundary S of real hyperbolic space.*

$$SO(n,1) = \{g \in SL(n+1, \mathbb{R}) : \langle gx, gx \rangle = \langle x, x \rangle \quad \forall x \in \mathbb{R}^{n+1}\}\,,$$

$$\langle x, x \rangle = \sum_1^n x_i^2 - x_{n+1}^2\,.$$

The Klein model of hyperbolic space is obtained by introducing inhomogeneous coordinates

$$z_i = \frac{x_i}{x_{n+1}} \qquad i = 1, \dots, n.$$

The invariant cone $\{x \in \mathbb{R}^{n+1} : \langle x, x\rangle < 0\}$ is mapped onto the unit ball

$$B = \{|z| < 1\} \subset \mathbb{R}^n$$

and its boundary is mapped onto the sphere

$$S = \partial B \subset \mathbb{R}^n.$$

B is the symmetric space

$$G/K = SO_0(n,1)/SO(n)$$

and S is the homogeneous space

$$G/MAN$$

with MAN a parabolic subgroup of $G = SO_0(n,1)$, which can be taken to be the stabilizer subgroup of the point $e_n = (0, \dots, 0, 1) \in S$

$$\begin{aligned} M &\cong SO(n-1) \\ A &\cong \mathbb{R}^* \\ N &\cong \mathbb{R}^{n-1}. \end{aligned}$$

The Lie algebra decomposes as

$$\mathbf{g} = \bar{\mathbf{n}} + \mathbf{a} + \mathbf{m} + \mathbf{n} = \bar{\mathbf{n}} + \mathbf{h}$$

with $\mathbf{h}$ the Lie algebra of MAN.

In this situation, $\mathrm{Ad}(h)$ maps $\bar{\mathbf{n}}$ into $\bar{\mathbf{n}} + \mathbf{a} + \mathbf{m}$, for all $h \in MAN$. Furthermore, since $\mathrm{Ad}(m)$ maps $\bar{\mathbf{n}}$ into itself (for all $m \in M$), the gradient construction is still an M−invariant construction, yet starting with a section f of a homogeneous vector bundle over MAN; the result, ∇f, will not be a section of such a bundle.

However it turns out, that after taking the projection onto an M−invariant subspace, the H−representation can be adjusted such that $pr \circ \nabla$ is in fact an equivariant differential operator.

This was discovered by Fegan [F]. In his work, Fegan gave the complete classification of the conformally equivariant first order differential operators. (The group G acts by conformal transformations on the sphere S).

1.3. Equivariant operators on homogeneous spaces. Left invariant differential operators on G are given by the Lie algebra vector fields $X \in g$:

$$Xf(g) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp tX).$$

Left translation by $a \in G$ is given by $g \to ag$. The induced transformation on functions is

$$\tau_a f(g) = f(a^{-1}g).$$

Left invariance means that

$$\tau_a(Xf) = X(\tau_a f) \qquad a \in G.$$

More general, let $\mathcal{U}(\mathbf{h})$ be the universal envelopping algebra of the Lie algebra $\mathbf{h}$ (of the closed subgroup $H \subset G$) and denote $Y \to Y'$ its principal automorphism induced by the Lie algebra automorphism $Y \to -Y$. Representations ρ of H on a vector space V induce representations ρ_* of $\mathbf{h}$ and of $\mathcal{U}(\mathbf{h})$ on V.

The tensor product of the moduli $\mathcal{U}(\mathbf{g})$ and $\mathrm{Hom}(V_1, V_2)$ over $\mathcal{U}(\mathbf{h})$ is the direct product modulo the equivalence relation determined by the linear span J of the elements

$$YU \otimes L - U \otimes (L \circ \rho_* Y')$$

with $U \in \mathcal{U}(\mathbf{g})$, $Y \in \mathbf{g}$, $L \in \mathrm{Hom}(V_1, V_2)$. This product is written

$$\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)\,.$$

Elements $U \otimes L \in \mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$ act on functions $f \in C^\infty(G \times_H V_1)$ by

$$(U \otimes L)f(g) = L(Uf)(g)\,.$$

The definition is clearly left invariant, it commutes with τ_a, $a \in G$.

The action is well defined, since the elements in J act trivially:

$$\begin{aligned}(YU \otimes L)f(g) &= L\left.\frac{d}{dt}\right|_{t=0} (Uf)(g \exp tY) \\ &= L\left.\frac{d}{dt}\right|_{t=0} \rho(\exp -tY)(U(f)(g)) \\ &= L\rho_*(Y')(Uf)(g) = (U \otimes L \circ \rho_*(Y'))f(g)\end{aligned}$$

□

Let $\mathbf{q}$ be a subspace of $\mathbf{g}$ complementary to $\mathbf{h}$ and let $\{Y_1, \dots, Y_q\}$ be a basis of $\mathbf{q}$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_q)$ of natural numbers write Y^α for $Y_1^{\alpha_1} \dots Y_q^{\alpha_q}$.

Normal form.
Every element in $\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$ can uniquely be written in the normal form

$$\sum_\alpha Y^\alpha \otimes L_\alpha$$

with $L_\alpha \in \mathrm{Hom}(V_1, V_2)$.

(For a proof see [KR2], proposition 1.1.)

The elements in $\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$ describe all left invariant linear differential operators from $C^\infty(G \times_H V_1)$ to $C^\infty(G, V_2)$. A further condition is required which guarantees that these operators map into $C^\infty(G \times_H V_2)$.

The subgroup H acts on $\mathcal{U}(\mathbf{g})$ by the adjoint representation and on $\mathrm{Hom}(V_1, V_2)$ by

$$L \longrightarrow \rho_2(h) \circ L \circ \rho_1(h^{-1}) \qquad L \in \mathrm{Hom}(V_1, V_2).$$

The tensor product action leaves the subspace J invariant, hence H acts on the tensor product over $\mathcal{U}(\mathbf{h})$. The set of H-invariant elements is denoted by

$$(\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2))^H\,.$$

Definition. A G–equivariant differential operator D is a G–left invariant linear differential operator between homogeneous vector bundles

$$D : C^\infty(G \times_H V_1) \longrightarrow C^\infty(G \times_H V_2)\,.$$

Proposition 1. [KR2]. *The space of G–equivariant differential operators from $(G \times_H V_1)$ to $(G \times_H V_2)$ is isomorphic to*

$$(\mathcal{U}(\mathrm{g}) \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2))^H\,.$$

Various ways of looking at equivariant linear differential operators are known from the literature, e.g. N. Wallach: *Harmonic analysis on homogeneous space*, 1973. The present proposition is quite explicit. It is specially designed to suit our purposes.

The essence of the argument leading to the above proposition is contained in the following little calculation:

For $X \in g,\ L \in \mathrm{Hom}(V_1, V_2),\ f \in C^\infty(G \times_H V_1)$:

$$\begin{aligned}(X \otimes L)f(gh) &= L \left.\frac{d}{dt}\right|_{t=0} f(gh \exp tX) \\ &= L \left.\frac{d}{dt}\right|_{t=0} \rho_1(h^{-1}) f(g \exp t\mathrm{Ad}(h)X) \\ &= (\mathrm{Ad}(h)X \otimes L \circ \rho_1(h^{-1}))f(g)\,.\end{aligned}$$

This will be equal to $\rho_2(h^{-1})(X \otimes L)f(g)$ for all $f \in C^\infty(G \times_H V_1)$, $g \in G$ and $h \in H$ if and only if $X \otimes L$ is invariant under the action of H (modulo the kernel of the action of $U \otimes \mathrm{Hom}(V_1, V_2)$).

Proposition 2. [KR2] *The first order equivariant linear differential operators D factor through the gradient.*

The operator $D : C^\infty(G \otimes_H V_1) \to C^\infty(G \otimes_H V_2)$ has a normal form

$$D = \sum_j Y_j \otimes L_j + L_0$$

with $\{Y_j\}$ a basis in $\mathbf{q}$. Under the identification of $\mathrm{Hom}(V_1, V_2)$ with $V_1^* \otimes V_2$ the maps L_j take the form

$$L_j = \sum e_k^* \otimes f_{jk} \qquad f_{jk} \in V_2$$

with $\{e_k\}$ a basis of V_1 and $\{e_k^*\}$ the dual basis. Define the homomorphism U : $\mathrm{Hom}(\mathbf{q}, V_1) \to V_2$ by

$$U : Y_j^* \otimes e_k \longrightarrow f_{jk}$$

so that

$$D = U \circ \nabla + L$$

with

$$\nabla = \sum_{jk} Y_j \otimes e_k^* \otimes (Y_j^* \otimes e_k)\,.$$

Warning: In general, ∇ will not be an equivariant operator.

2. First order equivariant differential operators on boundaries of symmetric spaces

2.1. **Parabolic subgroups.** Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of the semisimple Lie algebra $\mathfrak{g}$ of G with Cartan involution ϑ. Choose a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and an ordering $<$ in the dual $\mathfrak{a}'$ of $\mathfrak{a}$. If Σ is the set of non vanishing restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ then the Lie algebra has the root space decomposition

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_0 + \sum_{\lambda\in\Sigma} \mathfrak{g}_\lambda \\ \mathfrak{g}_\lambda &= \{X \in \mathfrak{g} : [H,X] = \lambda(H)X \qquad \forall H \in \mathfrak{a}\}. \end{aligned}$$

The subspace

$$\mathfrak{n} = \sum_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$$

($\sum^+$ the set of positive roots) is a nilpotent subalgebra of $\mathfrak{g}$ and the Lie algebra $\mathfrak{g}$ has the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}.$$

In the root space decomposition the subspace $\mathfrak{g}_0$ decomposes into $\mathfrak{a} + \mathfrak{m}$ with $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Upon setting

$$\overline{\mathfrak{n}} = \sum_{\lambda<0} g_\lambda = \sum_{\lambda>0} \vartheta g_\lambda$$

the decomposition becomes

$$\mathfrak{g} = \overline{\mathfrak{n}} + \mathfrak{a} + \mathfrak{m} + \mathfrak{n}.$$

The groups A and N are the analytic subgroups (i.e. the connected Lie subgroups) of G with Lie algebra $\mathfrak{a}$ and $\mathfrak{n}$ respectively. M is the centralizer of A in K

$$M = \{k \in K : kak^{-1} = a \quad \forall a \in A\}.$$

Its Lie algebra is $\mathfrak{m}$.

The subgroup MAN is a minimal parabolic subgroup and $S = G/MAN$ is the maximal boundary of the symmetric space G/K.

Recall the example: Real hyperbolic space and its boundary.

2.2. **Representations of MAN.** The equivariant differential operators on

$$S = G/MAN$$

are linear differential operators mapping sections of a homogeneous vector bundle $G \times_{MAN} V$ onto sections of another bundle $G \times_{MAN} W$. These bundles are associated to representations (ρ, V) of the subgroup MAN.

At this point the restriction is made that the representations of MAN have to be irreducible. We will see, that for geometrically relevant representations this is not always the case. Up to now we are not able to handle the non irreducible case in full generality.

An irreducible representation (ρ, V) of MAN must be trivial when restricted to N (this follows e.g. from Lie's theorem) and scalar when restricted to A. It is therefore completely described by a character μ of A and an irreducible representation δ of M. We will write

$$\rho(a) = a^{\mu} I$$

where a^{μ} stands for $\exp \mu(H)$ if $a = \exp H$, $\mu \in \mathrm{a}'$, and

$$\rho(m) = \delta(m) \qquad m \in M .$$

For the irreducible representation (ρ, V) we use the notation $(\mu, \delta; V)$ or simply (μ, δ).

Of central importance in view of the gradient construction is the fact that $\mathrm{Ad}(m)$ for each $m \in M$ preserves the root space decomposition:

$$\mathrm{Ad}(m) : \mathrm{g}_{\alpha} \longrightarrow \mathrm{g}_{\alpha} \qquad \forall \alpha .$$

The representation of M on the complexified space $\mathrm{g}_{\alpha}^{\mathbf{C}}$ will be denoted by $(\mathrm{Ad}_{g_{\alpha}}, \mathrm{g}_{\alpha}^{\mathbf{C}})$.

Theorem 1. [KR2].
If (δ, V) is a complex irreducible unitary representation of M, then the tensor product $(\mathrm{Ad}_{g_{\alpha}} \otimes \delta,\ \mathrm{g}_{\alpha}^{\mathbf{C}} \otimes V)$ decomposes with multiplicities one:

$$\begin{aligned} \mathrm{Ad}_{g_{\alpha}} \otimes \delta &= \bigoplus_{\ell} m_{\ell}\, \delta_{\ell} \\ \mathrm{g}_{\alpha}^{\mathbf{C}} \otimes V &= \bigoplus_{\ell} V_{\ell}^{m_{\ell}} \end{aligned}$$

where the multiplicities m_{ℓ} are one.

2.3. **The characterisation of first order equivariant differential operators on boundaries of symmetric spaces.** Starting with an irreducible representation $(\mu, \delta; V)$ of MAN we choose a basis $\{e_k\}$ of V and denote the dual basis with $\{e_k^*\}$.

Also choose an orthonormal basis $\{Y_j\}$ of $g_{-\lambda}$ with respect to the positive definite form

$$(\,.\,,\,.\,) = -B_{\mathrm{g}}(\,.\,,\,\vartheta .)$$

(where B_{g} is the Killing form). Then $\{\vartheta Y_j\}$ will be an orthonormal basis in g_{λ}.

The gradient operator, followed by the projection pr_{ℓ} onto an irreducible subspace V_{ℓ} in the tensor product decomposition $\mathrm{g}_{\alpha}^{\mathbf{C}} \otimes V = \oplus_{\ell} V_{\ell}$ takes the form

$$pr_{\ell} \circ \nabla^{\lambda} = \sum_{jk} Y_j \otimes e_k^* \otimes pr_{\ell}(\vartheta Y_j \otimes e_k) .$$

In order that this operator be equivariant, the characters in the corresponding MAN–representations have to be chosen appropriately.

Observe that ∇^{λ} is a restricted gradient operator in the sense, that it is only the gradient with respect to a single root space $g_{-\lambda}$. This is the reason for using the superscript λ in the notation.

For an irreducible unitary M−representation the Casimir operator

$$C_\delta = -\sum_k (\delta_* Z)^2 \qquad \{Z_k\} \text{ orthonormal basis of } \mathbf{m}$$

acts as a scalar

$$C_\delta = c(\delta) I .$$

Recall that a root $\lambda \in \Sigma_+$ is simple, if it is not the sum of two positive roots.

In the following formula, $H_\lambda \in a$ is determined by $B_{\mathbf{g}}(H, H_\lambda) = \lambda(H) \ \ \forall H \in a$.

The following theorem, the main result in this lecture, is stated and proved in this form in [KR2].

Partial results in this direction were obtained by A. Čap, J. Slovák, V. Souček [CSS] (series of papers in the years 1997-2000). Furthermore, in the setting of arbitrary parabolic subgroups, such a theorem is given by J. Slovák and V. Souček: *"Invariant operators of the first order on manifolds with a given parabolic structure"*, preprint 2000 [SS]. The result is also discussed in Ørsted's paper [O].

Theorem 2. *Assume that λ is simple, that $(\mu, \delta; V)$ is an irreducible complex representation of MAN and that (δ_ℓ, V_ℓ) is an irreducible component of the decomposition of $(\mathrm{Ad}_{\mathbf{g}_\lambda} \otimes \delta,\ \mathbf{g}_\lambda{}^{\mathbf{C}} \otimes V)$ over M. If the representation of MAN on V_ℓ is $(\mu + \lambda, \delta_\ell)$ and if*

$$2\mu(H_\lambda) = c(\delta) + c(\mathrm{Ad}_{\mathbf{g}_\lambda}) - c(\delta_\ell)$$

then $pr_\ell \circ \nabla^\lambda$ is a G−equivariant operator $C^\infty(G \times_{MAN} V) \to C^\infty(G \times_{MAN} V_\ell)$. Conversely, any first order equivariant operator

$$D : C^\infty(G \times_{MAN} V) \to C^\infty(G \times_{MAN} W)$$

(with irreducible actions of MAN on V and W) is of the form $U \circ pr_\ell \nabla^\lambda$ with U a MAN−equivariant mapping $V_\ell \to W$ and with $pr_\ell \nabla^\lambda$ determined by the data above.

2.4. **Tensor product decompositions.** Assume that M is a connected compact Lie group and T a maximal connected abelian subgroup (Cartan subgroup). Denote its Lie algebra by $\mathbf{h}$ and set

$$\begin{aligned} \mathbf{h}_R &= i\mathbf{h} \\ \mathbf{h}_C &= \mathbf{h} + i\mathbf{h} = \mathbf{h}_R + i\mathbf{h}_R . \end{aligned}$$

Unitary representations (δ, V) of M induce Lie algebra representations $\delta_* : \mathbf{m} \to \mathrm{End}\, V$. For $H \in \mathbf{h}$ $\delta_* H$ is skew Hermitian and hence diagonalizable with imaginary eigenvalues. V decomposes into a sum of the eigenspaces

$$V_\lambda = \{v \in V : \delta_*(H)v = \lambda(H)v \qquad \forall H \in \mathbf{h}_C\} .$$

The eigenvalues λ are in $(\mathbf{h}_C)'$ (dual space) and restricted to $\mathbf{h}_R$ they are real. They are called the weights of the representation and the spaces V_λ are called the weight spaces.

The non vanishing weights of the adjoint representation (of M on $\mathbf{m}$) are the roots $\alpha \in \Delta$.

Fix an ordering and denote the positive roots by Δ^+, the set of simple roots by $\prod$. A weight λ on $\mathbf{h}_C$ which is real on $\mathbf{h}_R$ is dominant, if

$$\frac{2\langle\lambda,\alpha\rangle}{|\alpha|^2} \geq 0 \qquad \forall\alpha \in \Delta^+$$

it is (algebraically) integral, if

$$\frac{2\langle\lambda,\alpha\rangle}{|\alpha|^2} \in \mathbb{Z} \qquad \forall\alpha \in \Delta\,.$$

Any representation space is the direct sum of its weight spaces. The integral dominant weights α are in one-to-one correspondence with the (equivalence classes of) irreducible representations δ_α.

We now refer to a theorem in representation theory which goes back to [PRV]. For a direct proof see [JKR].

Theorem 3. *Consider any finite dimensional complex representation δ of the compact connected group M. Then the multiplicity of the representation δ_γ (with integral dominant weight γ) in the decomposition of $\delta \otimes \delta_\alpha$ cannot exceed the dimension of the $\gamma - \alpha$ weight space of the representation δ.*

In particular, if all weight spaces of the representation δ are one-dimensional, then $\delta \otimes \delta_\alpha$ decomposes with multiplicities one.

For minimal parabolic subgroups MAN, the compact group M need not be connected. For the cases where M is connected, theorem 1 follows from this result. It only remains to verify that all weight spaces in Ad_{g_α} are one dimensional:

The maximal abelian subalgebra $\mathbf{h} \subset \mathbf{m}$ extends to a Cartan algebra $\tilde{\mathbf{h}} = \mathbf{h} + \mathbf{a}$ of $\mathbf{g}$. The space $\mathbf{g}_\alpha^{\mathbf{C}}$ is the direct sum of the $\tilde{\mathbf{h}}$−root spaces, say $\tilde{\mathbf{g}}_\gamma$ such that $\gamma|_{\mathbf{a}} = \alpha$. But these spaces $\tilde{\mathbf{g}}_\gamma$ are known to be one-dimensional. The $\tilde{\mathbf{g}}_\gamma$ are also the $\mathbf{h}$−weight spaces of Ad_{g_α} : Each of the $\tilde{\mathbf{g}}_\gamma$ must belong to a different weight:

$$\gamma \neq \gamma' \text{ and } \gamma|_{\mathbf{a}} = \gamma'|_{\mathbf{a}} \text{ implies } \gamma|_{\mathbf{h}} \neq \gamma'|_{\mathbf{h}}\,.$$

The argument given here is taken from [JKR]. It extends to the situation of general parabolic subgroups. The original proof in [KR2] of the multiplicity one result in the context of minimal parabolic subgroups was based on a detailed study of the subgroup M (including the cases in which M is not connected).

2.5. **General parabolic subgroups.** A parabolic subgroup H of the semisimple Lie group of non compact type G is a closed subgroup, which contains a minimal parabolic subgroup MAN.

Example. $G = SL(n,\mathbb{R})$.

The minimal parabolic subgroup is (conjugate to) the subgroup of upper triangular matrices MAN. The (general) parabolic subgroups which contain MAN are the block upper triangular subgroups, i.e.

$$\begin{pmatrix} \star & & & \star \\ & \boxed{\star} & & \\ & & \boxed{\star} & \\ 0 & & & \star \end{pmatrix}$$

These subgroups H have a decomposition

$$H = M_s A_s N_s .$$

The group $M_s A_s$ is the block diagonal part. The members of M_s have determinant ± 1 in each block along the diagonal. The elements of A_s are positive scalar matrices in each block along the diagonal, and the elements of N are upper triangular and equal to the identity in each diagonal block.

The parabolic subgroups containing MAN are in one-to-one correspondence with the subsets $\prod_s$ of the set $\prod$ of simple (positive) roots of $(\mathbf{g} : \mathbf{a})$. (See e.g. A. Knapp, *Representation theory of semisimple groups,* p. 133.) They are obtained from $\prod_s$ as follows:

$$\begin{aligned}
\mathbf{a}_s &= \{X \in \mathbf{a} : \lambda(X) = 0 \ \ \forall \lambda \in \textstyle\prod_s\} \subseteq \mathbf{a} \\
\Gamma_s^+ &= \{\lambda \in \Sigma^+ : \lambda|_{\mathbf{a}_s} \not\equiv 0\} = \Sigma^+ \setminus \Delta_s \\
\Delta_s &= \{\alpha \in \Sigma : \alpha|_{\mathbf{a}_s} \equiv 0\} \\
\mathbf{n}_s &= \Sigma_{\lambda \in \Gamma_s^+} \mathbf{g}_\lambda \\
\mathbf{m}_s &= \mathbf{m} + \Sigma_{\alpha \in \Delta_s} \mathbf{g}_\alpha .
\end{aligned}$$

A_s and N_s are two analytic subgroups with Lie algebra $\mathbf{a}_s$ and $\mathbf{n}_s$ respectively, and M_s is the centralizer of A_s in K. The Lie algebra of M_s is $\mathbf{m}_s$.

For the parabolic subgroups $M_s A_s N_s$ there is a Langlands decomposition

$$\mathbf{g} = \mathbf{m}_s + \mathbf{a}_s + \mathbf{n}_s + \overline{\mathbf{n}}_s$$

with $\overline{\mathbf{n}}_s = \vartheta \mathbf{n}_s$.

The multiplicity one theorem and the characterisation of the first order equivariant differential operators were the main results formulated for the minimal parabolic groups. They both carry over verbatim to the general parabolic groups. There are some technical changes because e.g. the groups M need not be compact (with consequence for the definition of the Casimir operator). But over all the changes in the formulations and in the proofs are minor. A complete discussion is presented in [JKR].

As already mentioned, in independent work J. Slovak and V. Soucek proved the same characterisation theorem for equivariant first order differential operators [SS]. Proceeding by a case-by-case-analysis they established the multiplicity one result for the classical groups.

Partial results on equivariant differential operators have also been obtained by Ørstel [O]. In the following chapter we will have an occasion to come back to some related results contained in this paper.

2.6. Elements of the proof for the main theorem. First order differential operators have the normal form

$$D = \sum_j Y_j \otimes L_j + L_0$$

with $\{Y_j\}$ an orthonormal basis of $\overline{\mathfrak{n}}$ (which is chosen such that each Y_j belongs to some $\mathfrak{g}_{-\lambda}$). The elements L_j are in $\mathrm{Hom}(V, W) = V^* \otimes W$ and can be written as

$$L_j = \sum_k e_k^* \otimes f_{jk} \qquad f_{jk} \subset W$$

where $\{e_k\}$ is a basis in V with dual basis $\{e_k^*\}$. Defining $U : \mathrm{Hom}(\overline{\mathfrak{n}}, V) \to W$ by

$$U : \vartheta Y_j \otimes e_k \longrightarrow f_{jk}$$

the operator can be written as

$$D = U \circ \nabla + L_0$$

with ∇ the gradient:

$$\nabla = \sum_{jk} Y_j \otimes e_k^* \otimes (\vartheta Y_j \otimes e_k) .$$

1. Elements $a \in A$ act on V, W as scalars according to the characters μ, ν respectively. This is written a^μ on V, a^ν on W and similarly $a^{-\lambda}$ on $\mathfrak{g}_{-\lambda}$:

$$\begin{aligned} a \cdot D &= a \cdot \left(\sum_{jk} Y_j \otimes e_k^* \otimes f_{jk} + \sum_k I \otimes e_k^* \otimes f_{0k} \right) \\ &= a^{-\mu} a^\nu \sum_{jk} a^{-\lambda(j)} Y \otimes e_k^* \otimes f_{jk} + a^{-\mu} a^\nu \sum I \otimes e_k^* \otimes f_{0k} . \end{aligned}$$

   The operator $D - a \cdot D$ is in normal form and has to vanish. This is possible

   either if $\nu = \mu$ and $f_{jk} = 0$ unless $j = 0$. This is the case of a zero order operator. It will not be considered any further.

   or if $\nu = \mu + \lambda$ for some root λ and $f_{jk} = 0$ unless $\lambda(j) = \lambda$.

   This case leads to $D = U \circ \nabla_\lambda$ with ∇_λ the gradient restricted to the root space $\mathfrak{g}_{-\lambda}$.
2. Equivariance under the action of $m \in M$ ensures that $U : V \to W$ will be an intertwining operator.
3. Invariance under the action of the subgroup N is equivalent to invariance under the induced action of $\mathfrak{n}$ (since N is connected). On V and on W the action is trivial. It remains to verify that for all ϑY_i in simple root spaces

$$(\vartheta Y_i) D = \sum_{\lambda(j)=\lambda} \sum_k [\vartheta Y_i, Y_j] \otimes e_k^* \otimes U(\vartheta Y_j \otimes e_k) = 0 .$$

   It can be concluded that λ must be a simple root, and thus $[\vartheta Y_i, Y_j] \in \mathfrak{a} + \mathfrak{m}$ for all indices i, j which occur in the sum.

$$\begin{aligned}(\vartheta Y_i)\cdot D &= \sum_{\lambda(j)=\lambda}\sum_k [\vartheta Y_i, Y_j]\otimes e_k^*\otimes U(\vartheta Y_j\otimes e_k)\\ &\equiv \sum_{\lambda(j)=\lambda}\sum_k I\otimes\rho_*^\vee([\vartheta Y_i, Y_j])e_k^*\otimes U(\vartheta Y_j\otimes e_k)\,.\end{aligned}$$

The invariance condition becomes

$$\sum_{ijk} Y_i\otimes\rho_*^\vee([\vartheta Y_i, Y_j])e_k^*\otimes U(\vartheta Y_j\otimes e_k)=0$$

with summation such that $\lambda(i)=\lambda(j)=\lambda$ (the remaining brackets $[\vartheta Y_i, Y_j]$ vanish). This tensor product is interpreted as a mapping

$$A:\mathrm{g}_\lambda\otimes V\longrightarrow \mathrm{g}_\lambda\otimes V$$

followed by U.

A lengthy calculation gives

$$A=-\mu(H_\lambda)I_{\mathrm{g}_\lambda}\otimes I_V+\sum_k(\delta_*\otimes\mathrm{ad}_{\mathrm{g}_\lambda})(Z_k\otimes Z_k)$$

with $\{Z_k\}$ a basis for m.

Finally this mapping is expressed in terms of the Casimir operators of the representations δ, $\mathrm{Ad}_{\mathrm{g}_\lambda}$ and the representations δ_ℓ from the decomposition of $\delta\otimes\mathrm{Ad}_{\mathrm{g}_\lambda}=\oplus\delta_\ell$.

The basic proposition is:
Define $E_{ij}\in\mathrm{Hom}(\mathrm{g}_\lambda,\mathrm{g}_\lambda)$ by $X_i\to X_j,\ \ X_k\to 0\ \ k\neq i,\ \ (\{X_j\}$ basis in $\mathrm{g}_\lambda)$. If $\{Z_j\}$ is an orthonormal basis for m, then $\sum_{i\neq j}[\vartheta X_i, X_j]\otimes E_{ij}=\sum Z_k\otimes\mathrm{ad}_{\mathrm{g}_\lambda}Z_k$.

The mapping A is:

$$\begin{aligned}A &= \sum_{ij}\rho_*[\vartheta Y_i, Y_j]\otimes Y_i\otimes\vartheta Y_j\\ &= \sum_{ij}\rho_*[\vartheta Y_i, Y_j]\otimes E_{ij}\,.\end{aligned}$$

For the details, the reader is referred to [KR2] and [JKR].

3. The Poisson transform

Let $(\mu,\delta;V)$ be a representation of MAN and (ρ,W) a representation of K. If $\iota: V\to W$ is an $M-$equivariant homomorphism:

$$\rho(m)\circ\iota=\iota\circ\delta(m)\qquad\forall m\in M$$

then there is a Poisson transform

$$P: C^\infty(G\times_{MAN}V)\longrightarrow C^\infty(G\times_K W)\,.$$

It is defined by the formula

$$Pf(g) = \int_K \rho(k)\iota f(g\,h)d\,k$$

this transform is G–equivariant for the G–action by left translation.

Example. The M–equivariant map ι_ϑ defining Poisson transforms of vector fields and one-forms will be $\iota = I - \vartheta$ for both $\overline{\mathbf{n}} \to p$ and $\mathbf{n} \to p$. For $X \in \mathbf{g}$ set

$$(X^B f)(x) = \frac{d}{dt} f(\exp tX \cdot x) \qquad x \in B = G/K$$

and similarly X^S on $S = G/MAN$. Then $X^S \in C^\infty(G \times_{MAN} \overline{\mathbf{n}})$ and $X^B \in C^\infty(G \times_K \mathbf{p})$ are the projections of the right invariant vector fields to S and B respectively.

Proposition 3. [KR1]. *If $B = G/K$ is irreducible as a symmetric space, with rank r and dimension n, then*

$$P(X^S) = \frac{n-r}{n} X^B$$

for all $X \in \mathbf{g}$.

The symmetric space G/K is irreducible, if the representation Ad of K restricts to an irreducible representation on $\mathbf{p}$. The rank is the dimension of $\mathbf{a}$ in the decomposition $\mathbf{g} = \overline{\mathbf{n}} + \mathbf{a} + \mathbf{m} + \mathbf{n}$.

This example illustrates the basic feature of the Poisson transform as an intertwining operator.

In the remainder of this section we will now use the Poisson transform to interpret the equivariant operators D^S on the boundary $S = G/MAN$ as the "boundary differential operators" of equivariant operators D^B on the symmetric space $B = G/K$.

Definition. $D^S : C^\infty(G \times_{MAN} V) \to C^\infty(G \times_{MAN} V')$ is the boundary operator for $D^B : C^\infty(G \times_K W) \to C^\infty(G \times_K W')$ if there exist Poisson transformations P and P' such that the following diagram commutes up to a multiplicative constant

$$\begin{tikzcd}
C^\infty(G \times_{MAN} V) \arrow[r, "D^S"] \arrow[d, "P"] & C^\infty(G \times_{MAN} V') \arrow[d, "P'"] \\
C^\infty(G \times_K W) \arrow[r, "D^B"] & C^\infty(G \times_K W).
\end{tikzcd}$$

We start with the situation that $D^S = pr_\ell \nabla^\lambda$ is an equivariant differential operator as described in the main result of chapter 2. Recall that $(\mu, \delta; V)$ is an irreducible representation of MAN and that the representation on the irreducible component V_ℓ in $\mathbf{g}^P_\lambda \otimes V$ is given by $(\mu + \lambda, \delta_\ell)$.

The operator

$$D^S : pr_\ell \nabla^\lambda : C^\infty(G \times_{MAN} V) \longrightarrow C^\infty(G \times_{MAN} V_\ell)$$

is then G–equivariant, given that λ is a simple root and that the equation

$$2\mu(H_\lambda) = c(\delta) + c(\mathrm{Adg}_\lambda) - c(\delta_\ell)$$

holds.

Suppose now that (ρ, W) is an irreducible representation of K and

$$\iota : V \longrightarrow W$$

an M–equivariant homeomorphism with associated Poisson transform P. Then

$$\iota_\vartheta \otimes \iota : \mathbf{g}_\lambda^{\mathbf{C}} \otimes V \longrightarrow \mathbf{p}^{\mathbf{C}} \otimes W$$

is M–equivariant. We assume now that there exists a K–invariant component $(\rho_j W_j)$ of $(\mathrm{Ad} \otimes \rho,\ \mathbf{p}^{\mathbf{C}} \otimes W)$ such that the following diagram commutes

$$\begin{tikzcd} V_\ell \arrow[r] \arrow[d, "\iota'"] & \mathbf{n}_\lambda^{\mathbf{C}} \otimes V \arrow[d, "\iota_\vartheta \otimes \iota"] \\ W_j \arrow[r] & \mathbf{p}^{\mathbf{C}} \otimes W \end{tikzcd}$$

The horizontal arrows are the inclusion maps and ι' is the restriction of $\iota_\vartheta \otimes \iota$ to V_ℓ. The Poisson integral obtained from ι' will be denoted by P'.

Theorem 4 (Ørsted). *With the above ons, D^S is the boundary differential operator of*

$$D^B = pr_j \circ \nabla$$

with ∇ the gradient operator of the symmetric space and pr_j the projection $\mathbf{p}^{\mathbf{C}} \otimes W \to W_j$.

For a proof we refer to [O]. □

4. Complex hyperbolic space

4.1. **The boundary of complex hyperbolic space.** Complex hyperbolic space is the symmetric space G/K with $G = SU(n+1,1)$ and $K = S(U(n+1) \times U(1))$. As a linear group G consists of the linear transformations $g \in SL(n+2, \mathbb{C})$ which preserve the hermition form

$$\langle \zeta, \eta \rangle = \sum_0^n \zeta_j \overline{\eta}_j - \zeta_{n+1} \overline{\eta}_{n+1} \,.$$

ζ acts as a transformation group on

$$B = \{ z \in \mathbb{C}^{n+1} : |z| < 1 \}$$

and on its boundary $\Sigma = \partial B$. This action is described by introducing inhomogeneous coordinates

$$z_j = \frac{\zeta_j}{\zeta_{n+1}} \qquad 0 \le j \le n \,.$$

The invariant cone $\langle \zeta, \zeta \rangle < 0$ is mapped onto B, and the boundary of the cone onto $S = \partial B$.

On the Lie algebra level $\mathbf{g} = \mathbf{k} + \mathbf{p}$ is given as

$$\mathbf{k} = \left\{ \left(\begin{array}{c|c} \star & \\ \hline & \star \end{array} \right) \quad \text{anti hermitian} \right\}$$

$$\mathbf{p} = \left\{ \left(\begin{array}{c|c} 0 & \star \\ \hline \star & 0 \end{array} \right) \quad \text{hermitian} \right\}.$$

The root space decomposition is

$$\begin{aligned} \mathbf{g} &= \mathbf{g}_{-2\alpha} + \mathbf{g}_{-\alpha} + \mathbf{a} + \mathbf{m} + \mathbf{g}_{\alpha} + \mathbf{g}_{2\alpha} \\ &= \overline{\mathbf{n}} + \mathbf{a} + \mathbf{m} + \mathbf{n}. \end{aligned}$$

$\mathbf{a}$ is 1-dimensional (G/K is a symmetric space of real rank one)

$$\mathbf{a} = \mathbb{R}H \quad \text{with } H = \left(\begin{array}{c|c|c} & & 1 \\ \hline & & \\ \hline 1 & & \end{array} \right) \in \mathbf{p}$$

$$\mathbf{m} = \left\{ \left(\begin{array}{c|c|c} \star & & \\ \hline & \star & \\ \hline & & \star \end{array} \right) \quad \text{anti hermitian} \right\} \subset \mathbf{k}$$

$$\mathbf{g}_\alpha = \operatorname{span}\{Y_1, \dots, Y_{2n}\}$$

with

$$Y_j = \frac{1}{2} \left(\begin{array}{c|c|c} & -e_j^* & \\ \hline e_j & & e_j \\ \hline & e_j^* & \end{array} \right) \qquad e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ j = 1, \dots, n$$

$$Y_{n+j} = \frac{i}{2} \left(\begin{array}{c|c|c} & e_j^* & \\ \hline e_j & & e_j \\ \hline & -e_j^* & \end{array} \right)$$

and

$$\mathbf{g}_{2\alpha} = \mathbb{R}Y_{2n+1} \quad \text{with } Y_{2n+1} = \frac{i}{2} \left(\begin{array}{c|c|c} 1 & & 1 \\ \hline & & \\ \hline -1 & & 1 \end{array} \right).$$

The Cartan involution ϑ on $\mathbf{g}$ is

$$\vartheta X = -X^*$$

and $\mathbf{g} = \mathbf{k} + \mathbf{p}$ is the decomposition of g into the ± 1 eigenspaces for ϑ.

A basis for $\mathbf{p}$ (orthogonal with respect to the Killing form) is $\{X_0, \dots, X_{2n+1}\}$ with

$$\begin{aligned} X_0 &= H \\ X_j &= (1-\vartheta)Y_j \qquad j = 1, \dots, 2n+1 . \end{aligned}$$

At the origin $0 = eK$ of the symmetric space $B = G/K$ the elements $X_j \in p$ can be identified with

$$\begin{aligned} X_j &\sim \frac{\partial}{\partial x_j} \qquad j = 1, \dots, n \\ X_{n+j} &\sim \frac{\partial}{\partial y_j} \qquad j = 1, \dots, n \\ X_0 &\sim \frac{\partial}{\partial x_0} \\ X_{2n+1} &\sim \frac{\partial}{\partial y_0} \end{aligned}$$

Geometric description of $G/K = B \subset \mathbb{C}^{n+1}$

Take $\rho(z) = \frac{1}{4}(1-|z|^2)$, $z \in \mathbb{C}^{n+1}$ as the defining function for the unit ball $B \subset \mathbb{C}^{n+1}$. Then

$$\Omega = -\frac{i}{2}\,\partial\overline{\partial}\log\frac{1}{\rho(z)}$$

defines a Kähler form on the tangent space TB to the complex hyperbolic space B. The corresponding Kähler metric (Riemannian metric) is

$$g(\,.\,,\,.\,) = -\Omega(\,.\,,\, J.\,) .$$

Here, J is the complex structure (multiplication by i):

$$\begin{aligned} J\frac{\partial}{\partial x_k} &= \frac{\partial}{\partial y_k} \\ J\frac{\partial}{\partial y_k} &= -\frac{\partial}{\partial x_k} . \end{aligned}$$

On the boundary $S = \partial B$ set $\vartheta = Jd\rho$. The horizontal tangent space $HS \subset TS$ is given by $\{X \in TS : \vartheta(X) = 0\}$ and the Levi form on HS is

$$g_0(\,.\,,\,.\,) = d\vartheta(\,.\,,\, J.\,) .$$

Observe that HS is the maximal J invariant subbundle of TS. The contact form ϑ determines the Reeb vector field $T \in TS$ through the equations

$$\vartheta(T) = 1 \quad \text{and} \quad T \lrcorner d\vartheta = 0 .$$

The tangent space TS is identified with the bundle $G \times_{MAN} \overline{\mathfrak{n}}$ and the horizontal space HS corresponds to the subbundle $G \times_{MAN} \mathfrak{g}_{-\alpha}$. On $G \times_{MAN} \overline{\mathfrak{n}}$ the group MAN

acts by $pr_{\overline{\mathfrak{n}}} \circ \mathrm{Ad}$. It is at this point that the difficulties arise: the action $pr_{\overline{\mathfrak{n}}} \circ \mathrm{Ad}$ is not irreducible. In fact, for $X \in \mathfrak{n}$

$$\mathrm{ad}X : \mathfrak{g}_{-\alpha} \longrightarrow \mathfrak{a} + \mathfrak{m} + \mathfrak{n},$$

but this is no longer true for $\mathfrak{g}_{-2\alpha}$. When $X \in \mathfrak{g}_{\alpha}$, then

$$\mathrm{ad}X : \mathfrak{g}_{-2\alpha} \longrightarrow \mathfrak{g}_{-\alpha}\,.$$

On the other hand, the action of MAN by $pr_{\mathfrak{g}_{-\alpha}} \circ \mathrm{Ad}$ is an irreducible action on $\mathfrak{g}_{-\alpha}$. Therefore the horizontal bundle $HS \sim G \times_{MAN} \mathfrak{g}_{-\alpha}$ behaves much better. It is the natural space in our setting.

The bundle of contact forms (i.e. differential 1− forms α on TS with $\ker \alpha = HS$ is the dual of the factor bundle TS/HS). Its fiber is identified with

$$(\overline{\mathfrak{n}}/\mathfrak{g}_{-\alpha})' = \mathfrak{g}_{2\alpha}\,.$$

This is a real line bundle wich will be denoted by $L_{2\alpha}$.

4.2. **Contact deformations: where an invariant differential operator appears.** A contact transformation $\varphi : S \to S$ is a diffeomorphism which preserves the contact structure (the horizontal bundle HS):

$$\varphi_* HS = HS\,.$$

The group actions are contact transformations: $g \in G$ acts on $S = G/MAN$ by left translations. The identification of HS with $G \times_{MAN} \mathfrak{g}_{-\alpha}$ shows, that HS is preserved under left translations. A contact deformation V is a vector field on S, which generates a flow of contact mappings φ_t. It is well known (theorem of P. Lieberman) that contact deformations can be obtained from functions $p : S \to \mathbb{R}$ by a first order differential operator D^S:

$$V = D^S p = pT + J\mathrm{grad}_0 p$$

where T is the Reeb vector field and $\mathrm{grad}_0 p$ the horizontal gradient of p, i.e. the horizontal field determined by

$$Xp = g_0(X, \mathrm{grad}_0 p) \ \text{ for all } X \in HS\,.$$

It turns out that D^S has an interpretation as an equivariant differential operator

$$D^S : C^\infty(G \times_{MAN} L_{-2\alpha}) \longrightarrow C^\infty(G \times_{MAN} \overline{\mathfrak{n}})\,.$$

In particular, the "function" p has to be interpreted as a section in the line bundle $G \times_{MAN} L_{-2\alpha}$ (dual to $G \times_{MAN} L_{2\alpha}$).

In our previous notation, the expression for D^S is

$$D^S = \sum_{j=1}^{n} (Y_j \otimes 1^* \otimes Y_{n+j} - Y_{n+j} \otimes 1^* \otimes Y_j) + I \otimes 1^* \otimes Y_{2n+1}\,.$$

D^S is the sum of a first order and a zero order differential operator. The first order part of the operator is

$$J \circ \nabla^\alpha$$

where ∇^α is the gradient operator with respect to $\mathfrak{g}_{-\alpha}$, and $\{Y_1, \dots, Y_{2n}\}$ is an orthogonal basis of $\mathfrak{g}_{-\alpha}$.

The operator J commutes with the action of M, since M acts on $\mathfrak{g}_{-\alpha}$ by unitary transformations.

Let us inspect the N-invariance of D^S by looking at the induced action of $\mathfrak{n}$: For $\vartheta Y_i \in \mathfrak{n}$

$$\begin{aligned}(\vartheta Y_i)\cdot D^S &= \sum_{j=1}^{n}([\vartheta Y_i, Y_j]\otimes 1^*\otimes Y_{n+1} - [\vartheta Y_i, Y_{n+j}]\otimes 1^*\otimes Y_j) \\ &+ I\otimes 1^*\otimes pr_{\mathfrak{g}_{-\alpha}}([\vartheta_i, Y_{2n+1}])\,.\end{aligned}$$

We need the following facts:

$$[\vartheta Y_i, Y_j]\in \mathfrak{m} \qquad \text{for } i\neq j$$

(and $\mathfrak{m}$ acts trivially on $L_{2\alpha}$).

$$\begin{aligned}[\vartheta Y_i, Y_i] &= -\frac{1}{2}H \quad (=[\vartheta Y_{n+i}, Y_j]) \qquad \text{and} \\ H\cdot 1^* &= 2\cdot 1^*.\end{aligned}$$

$$\begin{aligned}[\vartheta Y_i, Y_{2n+1}] &= Y_{n+i} \qquad i=1,\dots,n \\ [\vartheta Y_{n+i}, Y_{2n+1}] &= -Y_i \qquad i=1,\dots,n.\end{aligned}$$

These facts give for $i=1,\dots,n$.

$$\begin{aligned}(\vartheta Y_i)D^S &= I\otimes([\vartheta Y_i, Y_i]\cdot 1^*\otimes Y_{n+i} + 1^*\otimes pr_{-\alpha}([\vartheta Y_i, Y_{2n+1}])) \\ &= 0\end{aligned}$$

(and similarly for $i=n+1,\dots,n$).

Let us again point out that D^S is not one of the operators considered in the second chapter, because it maps into $C^\infty(G\times_{MAN}\bar{\mathfrak{n}})$ and the bundle $G\times_{MAN}\bar{\mathfrak{n}}$ is constructed from the representation $pr_{\bar{\mathfrak{n}}}\circ \mathrm{Ad}$ of MAN, which is <u>not</u> an irreducible representation.

4.3. **The Ahlfors operator.** On complex hyperbolic space,the operators

$$\begin{array}{ll} D = J\circ\nabla & C^\infty(G\times_K \mathbf{R}) \longrightarrow C^\infty(G\times_K \mathfrak{p}) \\ S = pr_{(2,0)}\circ\nabla & C^\infty(G\times_K \mathfrak{p}) \longrightarrow C^\infty(G\times_K \mathfrak{p}_{[2,0]})\end{array}$$

are equivariant. The first operator describes taking the symplectic gradient (with respect to the Kähler form Ω) and the second operator is the Ahlfors operator (with respect to the Kähler metric g). We write $\mathfrak{p}_{[2,0]}$ for the space of symmetric 2-tensors of trace zero and $pr_{(2,0)}$ for the corresponding projection operator.

On the boundary $S=\partial B$, the operator

$$D^S : C^\infty(G\times_{MAN} L_{-2\alpha}) \longrightarrow C^\infty(G\times_{MAN}\bar{\mathfrak{n}})$$

is equivariant. It describes the construction of contact deformations.

One is tempted to define a variant of the Ahlfors operator acting on $C^\infty(G\times_{MAN}\bar{\mathfrak{n}})$. We were not able to construct such an equivariant operator. A main difficulty lies in the circumstance, that the representation of MAN on $\bar{\mathfrak{n}}$ is not irreducible.

It turns out that instead a second order equivariant operator

$$T^S : C^\infty(G \times_{MAN} L_{-2\alpha}) \longrightarrow C^\infty(G \times_{MAN} \mathfrak{g}_{-\alpha}^{[2,0]})$$

exists which plays the role of its counterpart $T = S \circ D$, on the symmetric space.

The operator T^S

Set

$$\varepsilon_{jk} = (\vartheta Y_j \otimes \vartheta Y_k + \vartheta Y_k \otimes \vartheta Y_j) \otimes 1 - \frac{\delta_{jk}}{n} \sum_{i=1}^{2n} \vartheta Y_i \otimes \vartheta Y_i \otimes 1$$

where 1 stands for the number 1 regarded as an element of the fibre $\mathbb{R}$ of $L_{-2\alpha}$. Then T^S is given by

$$\begin{aligned} T^S &= \sum_{j,k=1}^{n} ((Y_j Y_k - Y_{n+j} Y_{n+k}) \otimes 1^* \otimes (\varepsilon_{j,n+k} + \varepsilon_{k,n+j}) \\ &+ (Y_{n+j}, Y_k + Y_k Y_{n+j}) \otimes 1^* \otimes (\varepsilon_{n+j,n+k} - \varepsilon_{jk}) . \end{aligned}$$

$$\begin{aligned} S : C^\infty(G \times_{MAN} L_{-2\alpha}) &\xrightarrow{D^S} C^\infty(G \times_{MAN} \overline{\mathfrak{n}}) \\ &\xrightarrow{T^S} \dots\dots\dots\dots\dots\dots \; C^\infty(G \times_{MAN} \mathfrak{g}_{-\alpha}^{[2,0]}) \end{aligned}$$

$$B : C^\infty(G \times_K \mathbb{R}) \xrightarrow{D} C^\infty(G \times_K \mathfrak{p}) \xrightarrow{S} C^\infty(G \times_K \mathfrak{p}^{[2,0]}).$$

The following theorem, the main result from [KR1], says that D^S and T^S are the boundary differential operators of the equivariant operators D and T on the symmetric space G/K.

Theorem 5.

$$\begin{aligned} DP &= \frac{2(n+2)}{2n+1} P D^S \\ TP &= \frac{(n+2)(n+3)}{n(n+1)} P T^S \end{aligned}$$

The Poisson transform P on the left hand side of these equations is the usual Poisson transform. On the right hand side of the first equation P goes with $\iota = I - \vartheta : \overline{\mathfrak{n}} \to \mathfrak{p}$ and P on the right hand side of the second equation is determined by the restriction of $(I - \vartheta) \otimes (I - \vartheta)$ to $\mathfrak{g}_{-\alpha}^{[2,0]}$.

4.4. **Application: Symplectic and quasiconformal extension.** If $v \in C^\infty(G \times_{MAN} \overline{\mathfrak{n}})$ is a contact deformation, then $w = Pv$ is a Hamiltonian vector field (with respect to the symplectic structure Ω).

$$\begin{array}{ccc} p & \xrightarrow{D^S} & v \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle cP} \\ h & \xrightarrow{D} & w. \end{array}$$

It turns out that, up to a constant, the vector field v is the boundary value of the vector field w on B. This means, that the contact deformation v on the boundary S has a continuous extension kw (k constant) to a symplectic transformation: The contact flow extends to a Ω–symplectic flow.

Quasiconformal mappings $\varphi : M \to M$ on a metric space (M, d) are homeomorphisms which have bounded distortion:

$$H(x,r) := \frac{\max\limits_{d(x,y)=r} \quad d(f(x), f(y))}{\min\limits_{d(x,y)=r} \quad d(f(x), f(y))}.$$

There exists a constant K such that

$$H(x) := \limsup_{r\to 0} H(x,r) \leq K \qquad \text{for all } x \in M.$$

On complex hyperbolic space $B = G/K$ the metric d is the Riemannian metric defined by the invariant (Kähler) metric g.

On the boundary $S = G/MAN$ d is the Carnot-Caratheodory metric derived from the Levi metric on the horizontal space HS. The quasiconformal mappings here are generalised contact transforms with a uniform bound on the distortion.

Quasiconformal deformations are vector fields which generate one-parameter flows of quasiconformal mappings. The distortion condition on the vector field is the condition that the Ahlfors operator maps it into a bounded quantity. The relevant theorems in our context are:

A vector field w on a Riemannian manifold generates a flow of quasiconformal mappings φ_t if

$$\|Sw\|_\infty \leq c.$$

Here, S is the Ahlfors operator. For a proof of this result in the Euclidean case see [R1] and in the general case see [P].

A vector field v on $S = G/MAN$ (i.e. on the boundary of complex hyperbolic space) generates a flow of quasiconformal mappings ψ_t if $v = D^S p$ and if

$$\|T^S p\|_\infty \leq c$$

For a proof see [KR1] (the case of the Heisenberg group) and [R2] (the case of the boundary of a strictly pseudoconvex domain). The constant of quasiconformality $K(t)$ for φ_t respectively ψ_t is of exponential growth:

$$K(t) \leq e^{\text{const}\,|t|}.$$

From the commutativity of the diagram up to a multiplicative constant

$$\begin{tikzcd} p \arrow[rr, "T^S"] \arrow[d, "P"] & & T^S p \arrow[d, "P"] \\ h \arrow[r, "D"] & w = Dh \arrow[r, "S"] & Th \end{tikzcd}$$

it can be concluded that $Th \in L^\infty(G \times_K \mathfrak{p}^{[2,0]})$ whenever $T^S p \in L^\infty(G \times_{MAN} \mathfrak{g}_{-\alpha}^{[2,0]})$. Since we already know that the contact deformation $v = D^S p$ extends to a Hamiltonian vector field $kw = k\,Dh$ (k constant) it can now be concluded that the quasiconformal deformation $v = D^S p$ with $\|Tp\|_\infty \leq c$ extends to a quasiconformal deformation kw (with $\|Sw\|_\infty \leq c'$) on the complex hyperbolic space.

Corollary 4. [KR1] *Quasiconformal deformations on S extend to symplectic quasiconformal deformations in B.*

Since symplectic mappings preserve the volume element, it follows that symplectic quasiconformal mappings are quasiisometries (with respect to the Kähler metric g).

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