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ON DEFORMATIONS OF FINITE OPERATOR CALCULUS OF ROTA

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ABSTRACT. Finite Operator q-Calculus Extension of Gian-Carlo Rota Finite Operator Calculus is proposed. The extension relies among others on the notion of shift in the limit invariance of q-delta operators.

1. INTRODUCTION

The algebras to be q-deformed here are the algebras $\mathcal{F}$ of formal series which are in their turn isomorphic to the corresponding algebras $\sum$ of shift invariant operators. These algebras are introduced and basic facts about them are derived by Gian-Carlo Rota in [Rota 1] where the author develops formal aspects of the calculus of finite differences. The calculus is then treated as an algebraization of the reduced incidence algebra of Boolean algebra. "Upon replacing Boolean algebra by some other incidence algebra, other similar" calculi "are obtained" [Rota.1].

We make here the first characteristic step towards the complete development of "q-incidence algebras environment" for enumerative problems.

For the beginning we start with oscillator-like algebras generators corresponding to enumerative problems i.e. we start with delta operators and their duals.

2. DELTA - OPERATOR; THE NOTION AND EXAMPLES

Starting at first with motivating examples we are going to define a so called delta operator $\delta : P \longrightarrow P$; where P denotes the algebra of polynomials over a field F; char $F = 0$.

Examples

1. $$\begin{cases} \left(\frac{d}{dx}p_n\right)(x) - np_{n-1}(x) = 0\,; \\ p_0(x) = 1, \quad p_n(0) = 0; \quad n > 0\,. \end{cases}$$

The solution is unique. $p_n(x) = x^n; \quad n \geq 0$

The paper is in final form and no version of it will be submitted elsewhere.

2. $\triangle := E - id;\quad (E^{\alpha}\varphi)(x) = \varphi(x+\alpha);\quad E^1 = E;$

$$\begin{cases} \triangle(p_n)(x) - np_{n-1}(x) = 0; \\ p_0(x) = 1, \quad p_n(0) = 0; \quad n > 0. \end{cases}$$

$p_n(x) = x^{\underline{n}};\quad n \geq 0$; where $x^{\underline{n}} := x(x-1)\ldots(x-n+1)$. The solution is unique.

3. $\nabla := id - E^{-1} \equiv E^{-1}\triangle;$

$$\begin{cases} \nabla(p_n)(x) - np_{n-1}(x) = 0; \\ p_0(x) = 1, \quad p_n(0) = 0; \quad n > 0. \end{cases}$$

$p_n(x) = x^{\overline{n}};\quad n \geq 0$; where $x^{\overline{n}} := x(x+1)\ldots(x+n-1)$. The solution is unique.

4. $$\begin{cases} (\partial_q p_n)(x) - n_q p_{n-1}(x) = 0; \qquad np = p_n = (x+c)^n \\ p_0(x) = 1, \quad p_n(0) = 0; \quad n > 0. \end{cases}$$

$p_n(x) = x^n;\quad n \geq 0$. The solution is unique, where: $(\partial_q\varphi)(z) = \frac{\varphi(z)-\varphi(qz)}{(1-q)z}$; - is Hahn derivative [Hahn.1] and $n_q \equiv \frac{1-q^n}{1-q};\quad q \neq 1$. ∂_q - is not a delta operator as it lacks shift invariance. $\partial := \frac{d}{dx}, \triangle, \nabla, \leftarrow$ these are examples of the so called delta operators [Rota.1].

Definition 2.1. Let $T : P \to P$; be a linear operator; T is shift invariant iff

$$\forall\alpha \in F;\ [T; E^{\alpha}] = 0.$$

Definition 2.2.
a) Let $\delta : P \to P$; the linear operator δ is a **delta** operator iff δ is shift invariant and $\delta(id) = const \neq 0$.
b) Let $\delta_q : P \to P$; the linear operator δ_q is a **q-delta** operator iff $\lim_{q\to 1}\delta_q \equiv \delta$ is shift invariant and $\delta_q(id) = const \neq 0$

$\{p_n\}_{n\geq 0}$ from examples [1-3] are examples of basic polynomials for delta operators $\delta_q = \frac{d}{dx}, \triangle, \nabla$, while $\{p_n\}_{n\geq 0}$ from the examples 4 is an example of q-basic polynomial sequence for q-delta operator δ_q.

Definition 2.3. A polynomial sequence $\{p_n\}_{n\geq 0}$; $deg\ p_n = n$; such that
1) $p_0(x) = 1$, 2) $p_n(x) = 0$; $n \geq 0$, 3) $\delta_q p_n(x) = n_{\chi} p_{n-1}$; $n > 0$
is called the q-χ-basic polynomial sequence of the q-delta operator δ_q.

Definition 2.4. Let $\{p_n\}_{n\geq 0}$ be the q-basic polynomial sequence of the q-delta operator δ_q; we define then a dual to δ-operator a liner map $\hat{x}$; $\hat{x} : P \to P; \hat{x}p_n = p_{n+1};\quad n \geq 0$.

3. Elements of the Finite Operator q-Calculus

The objective of [Rota.1] was a unified theory of special polynomials. We extend this objective to encompass also correspondent q-deformed families of polynomials. The way to achieve this goal in [Rota.1] was exploiting the duality between the $\hat{x}$ & $\frac{d}{dx}$ the predecessors of the delta operator notion and its dual. The technique used and co-invented mostly by [Rota.1] is of the past century origin and it is the so called symbolic calculus.

In this section we shell refer all the time to [Rota.1] where a systematic development of formal aspect of the calculus of finite differences has been provided. Let as start with recalling notations and definitions.

Definition 3.1. With P we denote the algebra of all polynomials in $x \in F$; char $F = 0$.

Definition 3.2. A polynomial sequences $\{p_k\}_0^\infty$ is such a sequence that $deg\, p_k = k$.

Definition 3.3. With $\sum_q$ we denote the algebra of F-linear and "in the limit $q \to 1$" shift invariant operators i.e. $T_q \in \sum$ iff $\lim_{q\to 1}[T_q, E^\alpha] = 0 \quad \forall\alpha \in F$, where $(E^\alpha\varphi)(z) = \varphi(z+\alpha)$.

Observation 3.1. Let $\delta_q \in \sum_q$, then for every constant polynomial $a \in P$ we have

$$\lim_{q\to 1}\delta_q a = 0.$$

Proof obvious - by linearity ☐

Observation 3.2. If $p \in P$; $deg\, p = n$ then $\delta_q p_n \in P$; $deg\,(\delta_q p_n) = n-1$.

Proof goes like in [Rota.1]; "just replace" shift invariance by "limit shift invariance" and note that δ_q - as a linear operator is "coefficient blind operator". ☐

Proposition 3.1. *Every q-delta operator δ_q has the unique sequences of $q-\chi$-basic polynomials.*

Proof. For $n=0$ put $p_0(z) = 1$, for $n=1$ put $p_1(z) = \frac{z}{\delta_q(id)}$. Then inducing on n assume that $\{p_k(z)\}$ have been defined for $k < n$. From this inductive assumption we infer that p_n is defined uniquely. For that to see it is enough to notice that for any $p \in P$ $\deg p = n$; i.e. $p(z) = az^n + \sum_{k=0}^{n-1} c_k p_k(z)$ & $a \neq 0$; we have $\delta_q p(z) = a\delta_q z^n + \sum_{k=1}^{n-1} c_k k_q p_{k-1}(z)$ & $\deg \delta_q(z^n) = n-1$. Hence there exist a unique choice of constants $c_1, \ldots, c_{n-1}$ for which ($\chi = q$ - here) $\delta_q p = n_\chi p_{n-1}$. This determines $p \equiv p_n$ uniquely except for the constant term c_0 which is however determined uniquely by the condition $p_n(0) = 0$; $n > 0$. ☐

Let R denote any analytic or rational function such that $R(q^n) \underset{q\to 1}{\longrightarrow} n$. We may introduce an infinite family or R-basic polynomial sequences according to:

Definition 2.3'. A polynomial sequence $\{p_n\}_{n\geq 0}$; $deg\, p_n = n$; such that

1) $p_0(x) = 1$,
2) $p_n(0) = 0$; $n > 0$,
3) $\delta_q p_n = R(q^n)\, p_{n-1}$

is called the R-basic polynomial sequence of the q-delta operator δ_q.

Inspired by the predecessors $\widehat{x}$ & $\frac{d}{dx}$ of the notions developed in [Rota.1] we introduce:

Definition 3.4. A polynomial sequences $\{p_n\}_0^\infty$ is of q-binomial type if it satisfies the recurrence

$$p_n(x+y) = \sum_{k\geq 0}\binom{n}{k}_q p_k(x)\, p_{n-k}(y); \quad \text{where} \quad \binom{n}{k}_q \equiv \frac{n_q^{\underline{k}}}{k_q!}.$$

Theorem 3.1. *$\{p_n\}_0^\infty$ is a q-basic sequence of some q-delta operator δ_q iff it is a sequence of q-binomial type.*

Proof. See [Rota.1] – and use limit shift invariance instead of shift invariance. ☐

Theorem 3.2. *Let T_q be - in the limit - a shift invariant operator. Let δ_q be a q-delta operator with q-basic sequence $\{p_n\}_o^\infty$ of its polynomials. Then* $T_q = \sum_{n\geq 0} \frac{a_n}{n_q!}\delta_q^n$; *where* $a_k = [T_q p_k(z)]_{z=0}$.

Proof goes like in [Rota.1] as no new explicit use of shift (limit shift) invariance is used. □

Theorem 3.3. *Let δ_q be a q-delta operator and let F_q be the algebra of formal* $\exp_q$ *series of the same field* $\mathbf{F}$ *for which δ_q is defined. Then there exists an isomorphism* $\varphi : \varphi : F_q \to \sum_q$ *of the algebra F_q onto the algebra $\sum_q$ of in the limit shift invariant operators which carriers*

$$f_q(t) = \sum_{k\geq 0} \frac{a_k t^k}{k_q!} \xrightarrow{\text{into}} T_q = \sum_{k\geq 0} \frac{a_k}{k_q!}\delta_q^k .$$

Proof. With obvious changes goes like in [Rota.1]. □

The generalization to R-labeled [Odzi.1], [Kwa.1] deformations is readily at hand (see the Definition (2.3') above). The extension towards incidence algebras also seams to be natural as stated by the main observation (see - next section). This observation constitutes the link with incident algebras.

4. Incidence algebras - possibility of q-extensions

Apart from Gian-Carlo Rota [Rota.1] the incidence algebras were independently discovered by H. Scheid [Sche.1] and D.A. Smith [Smith.1]; see also [Rota.2].

Definition 4.1. Let

$$I(P,\mathbf{F}) = \{f;\ f : PxP \to \mathbf{F}; f(x,y) = 0;\ \text{unless } x \prec y;\ x,y \in P\}$$

where $\mathbf{F}$ is a field; $char\mathbf{F} = 0$ and $(P,\prec)$ is locally finite partially ordered set. Then $(I(P,\mathbf{F}),\mathbf{F};+;*;\circ)$ is called the incidence algebra, where "+" & "∘" denote the sum of functions and usual multiplication by scalars, while for $f*g \in I(P,\mathbf{F})$

$$(f*g)(x,y) = \sum_{z\in P} f(x,z)\,g(z,y) \tag{4.1}$$

Recall that a partially order set is locally finite iff its every segment $[x,y] = \{z \in P; x \leq z \leq y\}$ is finite, hence the summation in (4.1) ranges over the finite segment $[x,y]$.

The following examples are taken from [Rota.1].

Example 4.1. Let P be the set of nonnegative integers $P = \{0,1,2,3,4,5,6,7,...\}$ & $\prec\equiv\leq$ then $I(P,\mathbf{F}) = \{(a_{ij}), a_{ij} = 0,\ i < j\} \subsetneq M_\infty(\mathbf{F})$ i.e. $(I(P,\mathbf{F}),\mathbf{F};+;*;\circ)$ is represented by the algebra of upper triangular infinite matrices over field $\mathbf{F}$.

Example 4.2. The algebra of formal power series is isomorphic to incidence algebra $R(P)$; $(P;\prec) \equiv (P;\leq)$; $P \equiv N \cup \{0\}$. This isomorphism is given by the bijective correspondence φ

$$\sum_{n\geq 0} a_n z^n \xrightarrow{\varphi} f \equiv \left\{ f_{ij}; f_{ij} = \left\{ \begin{array}{ll} a_{i-j} & i \leq j : i,j \in P \\ 0 & \text{otherwise} \end{array} \right. \right\}$$

where for $f, g, h \in R(P)$ $h \equiv f * g$ corresponds to convolution of $\varphi^{-1}(f)$ & $\varphi^{-1}(g)$ i.e. $h(i,j) = \sum_{i\leq k\leq j} f(i,k)\, g(k,j) = \sum_{i\leq k\leq j} a_{k-i} b_{j-k} \equiv \sum_{r=0}^{n} a_r b_{n-r}$ after setting $r = k - i$ & $n = j - i$.

Example 4.3. The algebra of formal exponential power series is isomorphic to incidence algebra $R(L(S))$; where $L(S) = \{A; A \subset S; |A| < \infty\}$; S is countable and $(L(S); \subseteq)$ is partially ordered set. As a matter of fact $R(L(S))$ is the so called reduced incidence algebra of the poset $L(S)$ [Rota.1]. The isomorphism is given by the bijective correspondence φ:

$$F(z) \equiv \sum_{n\geq 0} \frac{a_n}{n!} z^n \xrightarrow{\varphi} f = \left\{ f(A,B) = \left\{ \begin{array}{l} a_{|B-A|}\ A \subseteq B \\ 0\ \ \text{otherwise} \end{array} \right. ; A, B \in L(S) \right\}$$

where note for $f, g, h \in R(L(S))$ $h = f * g$ corresponds to binomial convolution of $F \equiv \varphi^{-1}(f)$ & $\varphi^{-1}(g) \equiv G$ i.e. $(H \equiv \varphi^{-1}(h))$ for $H(z) \equiv \sum_{n\geq 0} \frac{c_n}{n!} z^n$ & $G(z) \equiv \sum_{n\geq 0} \frac{b_n}{n!} z^n$; $c_n = \sum_{k\geq 0} \binom{n}{k} a_k b_{n-k}$.

Using the incidence algebra technique - apart from new ones - one may arrive very simply at previously know result [Rota.1]. As a matter of fact these are the so called reduced incidence algebra technique that we have in mind. With "at the point" convergence one makes $I(P; \mathbf{F})$ to be a topological algebra [Rota.1]. Incident algebras characterize p.o. sets as:

Theorem 4.1. *Let P & Q be locally finite partially ordered sets. Let $I(P; \mathbf{F})$ & $I(Q; \mathbf{F})$ algebras be isomorphic. Then P & Q are isomorphic.*

Of the more frequent use are reduced incidence algebras and incidence coefficients. Reduced incidence algebras are obtained as quotients of incident algebras segments' families and an order compatible equivalence relation. They corresponds to formal series of various kind. The incident coefficients are generalization of the binomial coefficients [Rota.1].

Definition 4.2. Let $\sim$ denote an equivalence relation defined on the family $S(P)$ of segments of P; with P - locally finite partially ordered set. Let $f, g \in I(P; \mathbf{F})$ be such that for $[x,y], [u,v] \in S(P)$ & $[x,y] \sim [u,v]$; $f(x,y) = f(u,v)$ & $g(x,y) = g(u,v)$. If $(f * g)(x,y) = (f * g)(u,v)$ $\forall [x,y], [u,v]$; $[x,y] \sim [u,v]$ then the relation "$\sim$" is said to be order compactible.

Definition 4.3. Let P be a locally finite partially ordered set equipped with a compatible equivalence relation $\sim$ on $S(P)$. The set of all functions defined on $S(P)/\sim$ with the product defined below in Definition (4.3) - is called the reduced incidence algebra $R(p; \sim)$.

In order to define the product of $f : S(P)/\sim \to \mathbf{F}$ and $g : S(P)/\sim \to \mathbf{F}$ function referred to in the definition above let us consider denote by $\alpha, \beta \ldots$ the nonempty equivalence classes of segments of P i.e. $\alpha, \beta \cdots \in S(P)/\sim$ and let us call them [Rota.1] *types*.

Definition 4.4. $(Map(S(P)/\sim; \mathbf{F}), \mathbf{F}; +; *; \circ) \equiv R(P;\sim)$ is an algebra under the multiplication "$*$" defined as follows:

$$\begin{aligned} (Map(S(P)/\sim\ ;\mathbf{F}) \ni f, g \to h := f * g; \\ S(P)/\sim\ \ni \alpha \to h(\alpha) := \sum_{(\ldots)} \begin{bmatrix} \alpha \\ \beta, \gamma \end{bmatrix} f(\beta)\, g(\gamma), \end{aligned}$$

where the sum $\sum_{(\ldots)}$ ranges over all ordered pairs (β, γ) of all types and the brackets $\begin{bmatrix} \alpha \\ \beta, \gamma \end{bmatrix}$ are defined below.

Definition 4.5. $\begin{bmatrix} \alpha \\ \beta, \gamma \end{bmatrix}$:= the number of distinct elements z in a segment $[x, y]$ of type α and such elements z that $[x, z]$ is of type β while $[z, y]$ is of type γ.

One may prove [Rota.1] that the reduced incidence algebra $R(P;\sim)$ { i.e. the incidence algebra modulo $\sim$} is isomorphic to a subalgebra of incidence algebra of P.
It is our actual aim to study q-deformations of these reduced incidence algebras with the first step being done by:

The main observation. Algebra $\sum_q \approx F_q$ is an example of the algebra of formal q-exponential power series which is isomorphic to the reduced incidence algebra $R(L(S))$; the isomorphism φ is given by the bijective correspondence:

$$F_q(z) \equiv \sum_{n\geq 0} \frac{a_n}{n_q!} z^n \xrightarrow{q} f = \left\{ f(A,B) = \begin{cases} a_{|B-A|_q}; & A \leq B \\ 0 & \text{otherwise} \end{cases} ; A, B \in L(S) \right\}$$

where for $f, g, h \in R(L(S))$; $h := f * g$ corresponds to q-binomial convolution i.e. for

$$\begin{aligned} H_q(z) &= \sum_{n\geq 0} \frac{c_n}{n_q!} z^n \quad \& \quad G_q(z) \equiv \sum_{n\geq 0} \frac{b_n}{n_q!} z^n; \\ [n]\, W(z) &\equiv [n]\,(f * g)(z) \equiv c_n; \quad c_n = \sum_{k\geq 0} \binom{n}{k}_q a_k b_{n-k}. \end{aligned}$$

Proof goes like in the non-deformed case [Rota.1] □

The generalizations to R-labeled [Odzi.1], [Kwa.1] deformations is readily at hand as R-exponential power series - {R-rational function} are easily to invent due to obvious generalization:

$$\sum_{n=0}^{\infty}\frac{z^n}{n_q!}\underset{\text{generalize}}{\longrightarrow}\sum_{n=0}^{\infty}\frac{z^n}{R(q)\,R(q^2)\ldots R(q^n)}\to \text{which under the choice } R(x)=\frac{1-x}{1-q}$$

becomes $\exp_q\{z\}=\sum_{n=0}^{\infty}\frac{z^n}{n_q!}$ hence the crucial definition:

Definition 4.6.

$$\exp_R\{z\}\sum_{k=0}^{\infty}\frac{z^k}{R(q)\,R(q^2)\ldots R(q^n)}$$

Due to this - one arrives immediately to

The main generalized observation.

Algebra $\sum_R \approx F_R$ is an example of the algebra of formal R-exponential power series which is isomorphic to the R-deformed reduced incidence algebra $R(L(S))$; the isomorphism φ is given by the bijective correspondence:

$$F_R(z)\equiv\sum_{n\geq 0}\frac{a_n}{R(q^n)!}z^n \xrightarrow{\varphi} f=\left\{f(A,B)=\left\{\begin{array}{c} a_{R(q^{|B-A|})};A\leq B \\ 0 \quad \text{otherwise}\end{array}\right.;A,B\in L(S)\right\}$$

where for $f,g,h\in R(L(S))$; $h:=f*g$ corresponds to R-binomial convolution i.e. for

$$\begin{aligned} H_q(z) &= \sum_{n\geq 0}\frac{c_n}{R(q^n)!}z^n \quad \& \quad G_q(z)\equiv\sum_{n\geq 0}\frac{b_n}{R(q^n)!}z^n; \\ [n]\,W(z) &\equiv [n]\,(f*g)(z)\equiv c_n; \quad c_n=\sum_{k=0}^{n}\binom{n}{k}_R a_k b_{n-k}. \end{aligned}$$

where $\binom{n}{k}_R\equiv\frac{R(q^n)^{\underline{k}}}{R(q^n)!}$; for example with $R(x)=\frac{1-x}{1-q}$; $\binom{n}{k}_R\equiv\frac{n_q^{\underline{k}}}{k_q!}$.

Proof goes like in the non-deformed case [Rota.1] □

This is the good starting point for further investigations.

References

[Hahn.1] W. Hahn, *Uber orthogonal Polynomen die q-Differenzengleichungen*, Math. Nachr. **1949**, 4–34.

[Kwa.1] A. K. Kwaśniewski, A. Tereszkiewicz, E. Mioduszewska, *Uogólnienie analizy matematycznej poprzez q-deformacje: od Eduarda Heinego do algebr kwantowych*, Lecture by A.K.K. at V-ta Środowiskowa Konferencja Matematyczna; 11-15 November 1998 r. Katholic University of Lublin Press (**1999**).

[Odzi.1] A. Odzijewicz, Commun. Math. Phys. **192** (1998), 183–215.

[Rota.1] Gian-Carlo Rota, *Finite Operator Calculus*,Academic Press, Inc. (1975).

[Rota.2] Gian-Carlo Rota, *On the Foundations of Combinatorial Theory, I.Theorie of Mobius Functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **2** (1964), 340–368.

[Scheid.1] H. Scheid, R. Sivaramakrishnan, *Certain Classes of arithmetic functions and the operation of additive convolution*, J. Reine Angew. Math. **245** (1970), 201–207

[Smith.1] D. A. Smith, *Incident functions as generalized arithmetic functions I*, Duke Math. J. **34** (1967), 617-634.

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