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q - DIFFERENCE CONFORMAL INVARIANT OPERATORS AND EQUATIONS *

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1. Introduction and summary of the classical case

1.1. Consider a Lie group G, e.g., the Lorentz, Poincaré, conformal groups, and differential equations

$$\mathcal{I} f \; = \; j \tag{1.1}$$

which are G-invariant. These play a very important role in the description of physical symmetries - recall, e.g., the examples of Dirac, Maxwell equations. It is important to construct systematically such equations for the setting of quantum groups (expected as q-difference equations). The hope is that these equations will have less singular behaviour than the classical counterparts.

The approach to this problem used here relies on the following : In the classical situation the differential operators $\mathcal{I}$ giving the equations above may be described as operators intertwining representations of complex and real semisimple Lie groups [38], [40], [59], [16].

To recall the notions, consider a semisimple Lie group G and two representations T, T' acting in the representation spaces C, C', which may be Hilbert, Fréchet, etc. An *intertwining operator* $\mathcal{I}$ for these two representations is a continuous linear map

$$\mathcal{I} \; : \; C \longrightarrow C' \tag{1.2}$$

such that

$$\mathcal{I} \circ T(g) \; = \; T'(g) \circ \mathcal{I} \; , \quad \forall g \in G \tag{1.3}$$

This is what precisely is meant when we say that the equation (1.1) is a G - *invariant equation*. Note that $\ker \mathcal{I}$, $\operatorname{im} \mathcal{I}$ are invariant subspaces of C, C', resp. If $\ker \mathcal{I} = 0$ and $\operatorname{im} \mathcal{I} = C'$, then the representations T and T' are called *equivalent*, otherwise T and T' are called *partially equivalent*. If $\ker \mathcal{I} \neq 0$, this means that the equation (1.1) with $j = 0$ has non-trivial solutions.

Such equations exist also for more general classes of Lie groups. However, if G is semisimple then there exist canonical ways for the construction of all intertwining operators and thus, of the G - invariant equations. [For simplicity we consider mostly semisimple Lie groups, though the same results are valid for reductive Lie groups, since only their semisimple subgroups are essential for these considerations.] These operators are of two types - differential and integral. For the integral intertwining operators, which we shall not discuss here, we refer to [38] for the mathematical side and to [27], [30], [28] for explicit examples and applications.

As stated we are interested in the invariant differential operators for which we refer to [40], [59], [16]. [For early examples and partial cases see, e.g., [31], [27], [36], [30], [48], [51], [15], [35], [7], [9], [57], [33], [34], [2], [3]. Note that we do not discuss here nonlinear invariant operators; for two different approaches to those we refer to [4](and references therein) and to [10].] There are many ways to find such operators, however, most of these rely on constructions which are not yet available for quantum groups. Here we shall apply a procedure [16] which is rather algebraic and can be generalized almost straightforwardly to quantum groups. This procedure is recalled in the next subsection.

1.2. Here we sketch the procedure of [16] illustrating the general notions with the (double covering group of the) Minkowski conformal group $SU(2,2)$. Let G be a real semisimple Lie group. (We noted already that we restrict to semisimple groups for simplicity. For more technical simplicity one may assume that in addition G is linear and connected.) Let $\mathcal{G}$ be the Lie algebra of G. We shall use the so-called Bruhat decompositions of $\mathcal{G}$

$$\mathcal{G} \;=\; \mathcal{N}^+ \oplus \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}^- \tag{1.4}$$

(considered as direct sum of linear spaces), where $\mathcal{A}$ is a noncompact abelian subalgebra, $\mathcal{M}$ (a reductive Lie algebra) is the centralizer of $\mathcal{A}$ in $\mathcal{G}$ (mod $\mathcal{A}$), and $\mathcal{N}^+$, $\mathcal{N}^-$, resp., are nilpotent subalgebras forming the positive, negative, resp., root spaces of the restricted root system $(\mathcal{G},\mathcal{A})$. For the conformal group the subalgebras $\mathcal{N}^-$, $\mathcal{M}$, $\mathcal{A}$, $\mathcal{N}^+$, are the subalgebras of translations, Lorentz transformations, dilatations, special conformal transformations, resp.

In general, a real noncompact Lie algebra $\mathcal{G}$ has more than one Bruhat decomposition. This is standard material, cf., e.g., [8]. (It is explained also in [16], or in [20].) Note that $\mathcal{P} = \mathcal{P}^+ = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}^+$ are subalgebras of $\mathcal{G}$, the so-called *parabolic subalgebras.* (The subalgebras $\mathcal{P}^+$ and $\mathcal{P}^- = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}^-$ are conjugate under the Cartan involution.) The parabolic subalgebras with minimal dimension are called *minimal parabolic subalgebras* of $\mathcal{G}$. Let us denote by $\mathcal{P}_0$ a minimal parabolic subalgebra: $\mathcal{P}_0 = \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}_0$. The number of non-conjugate parabolic subalgebras (counting also the trivial case $\mathcal{P} = \mathcal{G} = \mathcal{M}$) is 2^{r_0}, $r_0 = \dim \mathcal{A}_0$. The group $SU(2,2)$ has three non-trivial non-conjugate parabolic subalgebras of dimensions $9, 10, 11$. With the above identification $\mathcal{P}^\pm$ are maximal conjugate parabolic subalgebras; $\mathcal{P}^-$ is called usually the Weyl algebra (comprising the Poincaré algebra and the dilatations).

Let us now introduce the corresponding subgroups of G. Let K denote the maximal compact subgroup of G, and let $\mathcal{K}$ denote the Lie algebra of K. Then we have the simply connected subgroups $A = \exp(\mathcal{A})$, $N^\pm = \exp(\mathcal{N}^\pm)$. Further, M is the centralizer of A in G (mod A). (M has the structure $M = M_d M_r$, where M_d is a finite group, M_r is reductive with the same Lie algebra $\mathcal{M}$ as M.) Then $P = MAN^+$ (and their conjugate MAN^-) are called parabolic subgroups of G.

The importance of the parabolic subgroups stems from the fact that the representations induced from them generate all (admissible) irreducible representations of G. In fact, it is enough to use only the so-called *cuspidal* parabolic subgroups, singled out by the condition that $\operatorname{rank} M = \operatorname{rank} M \cap K$; thus M has discrete series representations.

Let P be a cuspidal parabolic subgroup. Let μ fix a discrete series representation D^μ on the Hilbert space V_μ or the so-called limit of a discrete series representation (cf. [37]). Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$.

We call the induced representation $\chi = \operatorname{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G. (These are called *generalized principal series representations* (or limits thereof) in [37].)

Consider the space of functions

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{\nu(H)} \cdot D^\mu(m^{-1})\mathcal{F}(g)\} \qquad (1.5)$$

where $a = \exp(H)$, $H \in \mathcal{A}$. The special property of the functions of $\mathcal{C}_\chi$ is called *right covariance* [16] (or *equivariance*). It is well known that $\mathcal{C}_\chi$ can be thought of as the space of smooth sections of the homogeneous vector bundle (called also vector G-bundle) with base space G/P and fibre V_μ, (which is an associated bundle to the principal P-bundle with total space G).

Then the elementary representation (ER) $\mathcal{T}^\chi$ acts in $\mathcal{C}_\chi$, as the left regular representation (LRR), by:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g') , \quad g, g' \in G \qquad (1.6)$$

(In practice, the same induction is used with non-discrete series representations of M and also with non-cuspidal parabolic subgroups.) One can introduce in $\mathcal{C}_\chi$ a Fréchet space topology or complete it to a Hilbert space (cf. [37]). Finally, note that in order to obtain the invariant differential operators one may consider the infinitesimal versions of (1.5) and (1.6) (cf. the end of this subsection).

The ERs differ from the LRR (which is highly reducible) by the specific representation spaces $\mathcal{C}_\chi$. In contrast, the ERs are generically irreducible. The reducible ERs form a measure zero set in the space of the representation parameters μ, ν. (Reducibility here is topological in the sense that there exist nontrivial (closed) invariant subspace.) The irreducible components of the ERs (including the irreducible ERs) are called *subrepresentations.*

The importance of the elementary representations stems from the following result:

Theorem. [41], [39] Every irreducible admissible representation of a real connected semisimple Lie group G with finite centre is equivalent to a subrepresentation of an elementary representation of G.
Remark: Admissibility is a technical condition which is usually fulfilled in the physically interesting examples.

The other feature of the ERs which makes them important for our considerations is a *highest weight module* (HWM) structure associated with them. [It would be a *lowest* weight module structure, if we replace $N = N^+$ with N^-, as is actually done in [16].] For this we introduce the right action of $\mathcal{G}^{\mathbb{C}}$ (the complexification of $\mathcal{G}$) by the standard formula:

$$(\hat{X}\mathcal{F})(g) = \frac{d}{dt}\mathcal{F}(g\exp(tX))|_{t=0} \qquad (1.7)$$

where, $X \in \mathcal{G}^{\mathbb{C}}$, $\mathcal{F} \in \mathcal{C}_\chi$, $g \in G$, which is defined first for $X \in \mathcal{G}$ and then is extended to $\mathcal{G}^{\mathbb{C}}$ by linearity. Note that this action takes $\mathcal{F}$ out of $\mathcal{C}_\chi$ for some X but that is exactly why it is used for the construction of the intertwining differential operators.

We illustrate the highest weight module structure in the case of the minimal parabolic subalgebra. In that case M is compact and V_μ is finite dimensional.

Consider first the case when M is non-abelian. Let v_0 be the highest weight vector of V_μ. Now we can introduce $\mathbb{C}$-valued realization $\tilde{\mathcal{C}}_\chi$ of the space $\mathcal{C}_\chi$ by the formula:

$$\varphi(g) \equiv \langle v_0, \mathcal{F}(g) \rangle \tag{1.8}$$

where $\langle , \rangle$ is the M-invariant scalar product in V_μ. On these functions the right action of $\mathcal{G}^{\mathbb{C}}$ is defined by:

$$(\hat{X}\varphi)(g) \equiv \langle v_0, (\hat{X}\mathcal{F})(g) \rangle \tag{1.9}$$

If M is abelian or discrete then V_μ is one–dimensional and we set $\varphi = \mathcal{F}$. Part of the main result of [16] is:

Proposition. The functions of the $\mathbb{C}$-valued realization $\tilde{\mathcal{C}}_\chi$ of the ER $\mathcal{C}_\chi$ satisfy :

$$\hat{X}\varphi = \Lambda(X) \cdot \varphi , \quad X \in \mathcal{H}^{\mathbb{C}} \tag{1.10a}$$
$$\hat{X}\varphi = 0 , \quad X \in \mathcal{G}_+^{\mathbb{C}} \tag{1.10b}$$

where $\Lambda = \Lambda_\chi \in (\mathcal{H}^{\mathbb{C}})^*$ is built canonically from χ, [it contains all the information from χ, except about the character ϵ of the finite group M_d], $\mathcal{G}_+^{\mathbb{C}}$, $\mathcal{G}_-^{\mathbb{C}}$, are the positive, negative root spaces of $\mathcal{G}^{\mathbb{C}}$, i.e., we use the standard triangular decomposition $\mathcal{G}^{\mathbb{C}} = \mathcal{G}_+^{\mathbb{C}} \oplus \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}_-^{\mathbb{C}}$.

Now we note that conditions (1.10) are the defining conditions for the highest weight vector of a highest weight module (HWM) over $\mathcal{G}^{\mathbb{C}}$ with highest weight Λ. Moreover, special properties of a class of highest weight modules, namely, Verma modules, are immediately related with the construction of invariant differential operators.

To be more specific let us recall that a Verma module is a highest weight module V^Λ with highest weight Λ, such that $V^\Lambda \cong U(\mathcal{G}_-^{\mathbb{C}})v_0$, where v_0 is the highest weight vector, $U(\mathcal{G}_-^{\mathbb{C}})$ is the universal enveloping algebra of $\mathcal{G}_-^{\mathbb{C}}$. Verma modules have the following universality property: every HWM is isomorphic to a factor-module of the Verma module with the same highest weight.

Generically, Verma modules are irreducible, however, we shall be mostly interested in the reducible ones since these are relevant for the construction of differential equations. We recall the Bernstein-Gel'fand-Gel'fand [5] criterion according to which the Verma module V^Λ is reducible iff

$$2\langle \Lambda + \rho, \beta \rangle - m\langle \beta, \beta \rangle = 0 \tag{1.11}$$

holds for some $\beta \in \Delta^+$, $m \in \mathbb{N}$, where Δ^+ denotes the positive roots of the root system $(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})$, ρ is half the sum of the positive roots Δ^+.

Whenever (1.11) is fulfilled there exists [13] in V^Λ a unique vector v_s, called *singular vector*, such that $v_s \notin \mathbb{C}v_0$ and it has the properties (1.10) of a highest weight vector with shifted weight $\Lambda - m\beta$:

$$\hat{X}v_s = (\Lambda - m\beta)(X) \cdot v_s , \quad X \in \mathcal{H}^{\mathbb{C}} \tag{1.12a}$$
$$\hat{X}v_s = 0 , \quad X \in \mathcal{G}_+^{\mathbb{C}} \tag{1.12b}$$

The general structure of a singular vector is [16]:

$$v_s = P_{m\beta}(X_1^-, \ldots, X_\ell^-) v_0 \qquad (1.13)$$

where $P_{m\beta}$ is a homogeneous polynomial in its variables of degrees mk_i, where $k_i \in \mathbb{Z}_+$ come from the decomposition of β into simple roots: $\beta = \sum k_i \alpha_i$, $\alpha_i \in \Delta_S$, the system of simple roots, X_j^- are the root vectors for $-\alpha_j$, α_j are the simple roots, $\ell = \operatorname{rank} \mathcal{G}^{\mathbb{C}}$. It is obvious that (1.13) satisfies (1.12a), while conditions (1.12b) fix the coefficients of $P_{m\beta}$ up to an overall multiplicative nonzero constant.

Now we are in a position to define the intertwining differential operators, corresponding to the singular vectors.

Let the signature χ of an ER be such that the corresponding $\Lambda = \Lambda_\chi$ satisfies (1.11) for some $\beta \in \Delta^+$ and some $m \in \mathbb{N}$. [If β is a real root, (i.e., $\beta|_{\mathcal{H}_m^{\mathbb{C}}} = 0$, where $\mathcal{H}_m$ is the Cartan subalgebra of $\mathcal{M}$), then some conditions are imposed on the character ϵ representing the finite group M_d [55]]. Then there exists an intertwining differential operator [16]

$$\mathcal{D}_{m\beta} : \tilde{\mathcal{C}}_\chi \longrightarrow \tilde{\mathcal{C}}_{\chi'} \qquad (1.14)$$

where χ' is such that $\Lambda' = \Lambda_{\chi'} = \Lambda - m\beta$.

The important fact is that (1.14) is explicitly given by [16]

$$\mathcal{D}_{m\beta}\varphi(g) = P_{m\beta}(\hat{X}_1^-, \ldots, \hat{X}_\ell^-)\varphi(g) \qquad (1.15)$$

where $P_{m\beta}$ is the same polynomial as in (1.13) and $\hat{X}_j^-$ denotes the action (1.7). We stress that these are explicit and compact expressions once the singular vectors are known. The latter are known for $\mathcal{G}^{\mathbb{C}} = sl(n, \mathbb{C})$, and for a large class of positive roots for the other simple Lie algebras [19].

One important simplification is that in order to check the intertwining properties of the operator in (1.15) it is enough to work with the infinitesimal versions of (1.5) and (1.6), i.e., work with representations of the Lie algebra. Thus, also in the quantum group setting we work with representations of quantum algebras.

Naturally, the above Verma module constructions are related with the elementary representations of the complexification $G^{\mathbb{C}}$ of G, or infinitesimally, of $\mathcal{G}^{\mathbb{C}}$. The corresponding representation spaces (in particular, the right covariance conditions) are given by (instead of (1.5)) [58] :

$$\mathcal{C}_\Lambda = \{\varphi_c \in C^\infty(G^{\mathbb{C}}) \mid \varphi_c(ghn) = e^{\Lambda(H)} \cdot \varphi_c(g)\} \qquad (1.16)$$

where $g \in G^{\mathbb{C}}$, $h = \exp(H)$, $H \in \mathcal{H}^{\mathbb{C}}$, $n \in G_+^{\mathbb{C}} = \exp(\mathcal{G}_+^{\mathbb{C}})$, Λ is as in (1.10). Note also that $\varphi_c|_G = \varphi$, and (1.10) holds also for φ_c. Thus, below we shall use the notation φ also for these functions.

This finishes the sketch of the classical results in general. In the next subsection we present an example.

1.3. In this subsection we follow mostly [27]. In the setting of the previous subsection we take $G = SO_e(n+1, 1)$, the Euclidean conformal group of n-dimensional Euclidean space, and also its double covering group $\hat{G} = Spin(n+1, 1)$.

We take $n \geq 3$, while for $n = 1,2$ we refer to Appendix B of [27]. In this case $K = SO(n+1)$, $\hat{K} = Spin(n+1)$, $M = SO(n)$, $\hat{M} = Spin(n)$, $\dim A = 1$, $N^\pm \cong I\!R^n$. As in [27] we first take irreducible representations of M labelled by $\mu = (0,\dots,0,\ell)$ ([n/2] entries), $\ell \in Z\!\!Z_+$. An alternative labelling (differing by the values of the half-sum of the positive roots) is: $\mu = \{0,1,\dots,\tilde{h}-2,\ell+\tilde{h}-1\}$, where $\tilde{h} \equiv [\frac{n}{2}]$. Usually these are realized as symmetric traceless tensors $\mathcal{F}_{\mu_1,\dots,\mu_\ell}$ of rank ℓ, $\mu_j = 1,\dots,n$. However, they can be also realized in the space of homogeneous polynomials of degree ℓ

$$\varphi(\xi) \;=\; \mathcal{F}_{\mu_1,\dots,\mu_\ell}\, \xi_{\mu_1}\cdots\xi_{\mu_\ell} \tag{1.17}$$

on the complex light cone:

$$I\!K_n \;=\; \{\xi \in \mathbb{C}^n \mid \xi^2 \;\equiv\; \xi_1^2+\cdots+\xi_n^2 \;=\; 0\} \tag{1.18}$$

(summation over repeated indices is understood). Each such function has a unique harmonic (homogeneous polynomial) extension $\check{\varphi}(\zeta)$, $\zeta \in \mathbb{C}^n$:

$$\begin{aligned} \Delta_\zeta\, \check{\varphi}(\zeta) \;&=\; 0\,, \quad \Delta_\zeta \;\equiv\; \frac{\partial^2}{\partial\zeta_1^2} + \cdots + \frac{\partial^2}{\partial\zeta_n^2} \\ \check{\varphi}(\zeta) \;&=\; \check{\mathcal{F}}_{\mu_1,\dots,\mu_\ell}\, \zeta_{\mu_1}\cdots\zeta_{\mu_\ell} \\ \check{\varphi}(\xi) \;&=\; \varphi(\xi)\,, \quad \xi \in I\!K^n \end{aligned} \tag{1.19}$$

Let us denote the signature of the class of ERs we consider by $\chi = [\ell, c]$ where $c \in \mathbb{C}$ determines the character of A. The reducible ERs from this class are parametrized by two integers $\ell \in Z\!\!Z_+$, $p \in I\!N$. They form four families with the following signatures:

$$\chi^\pm_{\ell p} \;=\; \left[\ell,\; \pm\left(\tfrac{n}{2}+\ell+p-1\right)\right] \tag{1.20a}$$

$$\chi'^\pm_{\ell p} \;=\; \left[\ell+p,\; \pm\left(\tfrac{n}{2}+\ell-1\right)\right] \tag{1.20b}$$

For fixed ℓ, p the four ERs $\chi^\pm_{\ell p}$, $\chi'^\pm_{\ell p}$ have the same values of the Casimir operators and are partially equivalent. Let us denote by $\tilde{\mathcal{C}}^\pm_{\ell p}$, $\tilde{\mathcal{C}}'^\pm_{\ell p}$ the representation spaces with signature $\chi^\pm_{\ell p}$, $\chi'^\pm_{\ell p}$, resp. The intertwining maps between $\tilde{\mathcal{C}}^+_{\ell p}$ and $\tilde{\mathcal{C}}^-_{\ell p}$ and between $\tilde{\mathcal{C}}'^+_{\ell p}$ and $\tilde{\mathcal{C}}'^-_{\ell p}$ are integral operators, cf. [27]. The intertwining operator between $\tilde{\mathcal{C}}^-_{\ell p}$ and $\tilde{\mathcal{C}}'^-_{\ell p}$ is given by:

$$\begin{aligned} &d^p \;:\; \tilde{\mathcal{C}}^-_{\ell p} \longrightarrow \tilde{\mathcal{C}}'^-_{\ell p} \\ &(d^p\, \varphi)(x;\xi) \;=\; (\xi\cdot\nabla)^p\, \varphi(x;\xi)\,, \qquad \varphi \in \tilde{\mathcal{C}}^-_{\ell p}\,, \\ &x \in I\!R^n, \quad \xi \in I\!K^n, \quad \xi\cdot\nabla \;=\; \xi_\mu \nabla_\mu\,, \quad \nabla_\mu \;=\; \frac{\partial}{\partial x_\mu} \end{aligned} \tag{1.21}$$

The intertwining operator between $\chi'^{+}_{\ell p}$ and $\chi^{+}_{\ell p}$ is given by:

$$\begin{aligned}
&d'^{p} \ : \ \tilde{\mathcal{C}}'^{+}_{\ell p} \longrightarrow \tilde{\mathcal{C}}^{+}_{\ell p} \\
&(d'^{p}\, \varphi)\,(x;\xi) \ = \ \left(D^{\xi}\cdot\nabla\right)^{p}\, \varphi(x;\xi)\ , \qquad \varphi \in \tilde{\mathcal{C}}'^{+}_{\ell p}\ , \\
&x \in I\!\!R^{n}, \quad \xi \in I\!\!K^{n}, \quad D^{\xi}\cdot\nabla \ = \ D^{\xi}_{\mu}\nabla_{\mu} \\
&D^{\xi}_{\mu} \ \equiv \ \left(\left(\tfrac{n}{2} - 1 + \xi\cdot\partial^{\xi}\right)\, \partial^{\xi}_{\mu} \ - \ \tfrac{1}{2}\, \xi_{\mu}\, \Delta_{\xi}\right)\ , \quad \xi\cdot\partial^{\xi} \ = \ \xi_{\mu}\partial^{\xi}_{\mu}\ , \quad \partial^{\xi}_{\mu} \ = \ \frac{\partial}{\partial\xi_{\mu}}
\end{aligned} \tag{1.22}$$

Note that the operator D^{ξ} (cf. Appendix A of [27], in particular, formula (A.47)) is an *interior operator* on the light cone $I\!\!K^{n}$, i.e., for any polynomial $\varphi(\xi)$ we have:

$$\left(D^{\xi}_{\mu}\xi^{2}\ \varphi(\xi)\right)_{|_{\xi^{2}=0}} \ = \ 0 \tag{1.23}$$

We finish this example by stressing that the intertwining differential operators d^{p} and d'^{p} are just powers of one and the *same expressions* in all cases irrespectively of the rank of the tensors they are acting on, which is in sharp contrast with the corresponding expressions if one is using the realization $\mathcal{F}_{\mu_1,\dots,\mu_\ell}$ with tensor indices.

If one is not restricting to symmetric traceless tensors of $M = SO(n)$ then there are other collections of ERs which have the same values of the Casimir operators and are partially equivalent. Such collections are called *multiplets* and for the general treatment we refer to [14]. Here we restrict to those multiplets of $G = SO(n+1,1)$, $\hat{G} = Spin(n+1,1)$, resp. which are in 1-to-1 correspondence with the (finite-dimensional) unitary irreducible representations (UIRs) of $SO(n+2)$, $Spin(n+2)$, resp. (the compact form of the complexification of G, $\hat{G}$, resp.). [These multiplets are 'maximal' w.r.t. to the number of ERs they contain and they correspond 1-to-1 to the elements of the restricted Weyl group $W(G,A)$, cf. [14].] Let us parametrize the UIRs of $Spin(n+2)$ as follows:

$$\begin{aligned}
\tau \ &= \ \{m_1\,,\dots,\, m_{\tilde{h}+1}\}\,, \quad n \text{ even}\,, \quad m_j \in Z\!\!Z/2\,, \quad |m_1| < m_2 < \cdots < m_{\tilde{h}+1} \\
\tau \ &= \ \{m_1\,,\dots,\, m_{\tilde{h}+1}\}\,, \quad n \text{ odd}\,, \quad m_j \in Z\!\!Z/2\,, \quad 0 \le m_1 < m_2 < \cdots < m_{\tilde{h}+1}
\end{aligned} \tag{1.24}$$

($\tilde{h} \ \equiv \ [\frac{n}{2}]$). To this UIR corresponds a multiplet of exactly $2\tilde{h}+2$ ERs of $\hat{G}$ which have the same values of the Casimir operators and are partially equivalent. Moreover, there are no other ERs partially equivalent to those (cf., e.g., [14]). In general, the signature of the ERs of $\hat{G}$ are labelled as follows:

$$\chi \ = \ \{m_1\,,\dots,\, m_{\tilde{h}}\,;\, c\} \tag{1.25}$$

where the first $\tilde{h}$ entries are the labels of the UIRs of $\hat{M} = Spin(n)$, and the last entry labels (as above) the characters of A. (In this notation the ERs induced from symmetric traceless tensor representations of M are labelled as $\{0,1,\dots,\ell+\tilde{h}-1;c\}$.) Accordingly the signatures of the ERs in the multiplet under consideration are labelled as:

$$\begin{aligned}
\chi_1^\pm &= \{\epsilon m_1, \ldots, m_{\tilde h}\,;\, \pm m_{\tilde h+1}\} \\
\chi_2^\pm &= \{\epsilon m_1, \ldots, m_{\tilde h-1}, m_{\tilde h+1}\,;\, \pm m_{\tilde h}\} \\
\chi_3^\pm &= \{\epsilon m_1, \ldots, m_{\tilde h-2}, m_{\tilde h}, m_{\tilde h+1}\,;\, \pm m_{\tilde h-1}\} \\
&\ldots \\
\chi_{\tilde h}^\pm &= \{\epsilon m_1, m_3, \ldots, m_{\tilde h}, m_{\tilde h+1}\,;\, \pm m_2\} \\
\chi_{\tilde h+1}^\pm &= \{\epsilon m_2, \ldots, m_{\tilde h}, m_{\tilde h+1}\,;\, \pm m_1\} \\
\epsilon &= \begin{cases} \pm, & \text{for } n \text{ even} \\ 1, & \text{for } n \text{ odd} \end{cases}
\end{aligned} \tag{1.26}$$

[The signatures in (1.20) $\chi_{\ell p}^\pm$, $\chi'^\pm_{\ell p}$, correspond to $\chi_1^\pm$, $\chi_2^\pm$, resp.] Note that in every multiplet only the ER χ_1^- has a finite-dimensional nonunitary subrepresentation. The latter has the same dimension as the fixed UIR of $Spin(n+2)$ in (1.24) to which the above multiplet corresponds. The ERs in the multiplet are related by intertwining integral operators and by $2\tilde h$ different intertwining differential operators. Let us denote by $\tilde{C}_i^\pm$ the representation space with signature $\chi_i^\pm$. The integral operators intertwine the pairs $\tilde{C}_i^+$ and $\tilde{C}_i^-$:

$$G_i^+ : \tilde{C}_i^- \longrightarrow \tilde{C}_i^+, \qquad G_i^- : \tilde{C}_i^+ \longrightarrow \tilde{C}_i^- \tag{1.27}$$

The intertwining differential operators act as follows:

$$\begin{aligned}
d_i &: \tilde{C}_i^- \longrightarrow \tilde{C}_{i+1}^-, \quad i = 1, \ldots, \tilde h, \quad \forall n \\
d'_i &: \tilde{C}_{i+1}^+ \longrightarrow \tilde{C}_i^+, \quad i = 1, \ldots, \tilde h, \quad \forall n \\
d_{\tilde h} &= d'_{\tilde h}, \quad n \text{ even} \\
d_{\tilde h+1} &: \tilde{C}_{\tilde h+1}^- \longrightarrow \tilde{C}_{\tilde h}^+, \quad n \text{ even} \\
d_{\tilde h+1} &: \tilde{C}_{\tilde h}^- \longrightarrow \tilde{C}_{\tilde h+1}^+, \quad n \text{ even}
\end{aligned} \tag{1.28}$$

The degrees of these intertwining differential operators are given just by the differences of the c entries:

$$\begin{aligned}
\deg d_i &= \deg d'_i = m_{\tilde h+2-i} - m_{\tilde h+1-i}, \quad i = 1, \ldots, \tilde h, \quad \forall n \\
\deg d_{\tilde h+1} &= m_2 + m_1, \quad n \text{ even}
\end{aligned} \tag{1.29}$$

The equalities between some intertwining differential operators for n even in (1.28) mean that these have the same expressions as actual differential operators. This is possible first of all because these operators are produced by singular vectors corresponding to the same positive roots of the root system of $so(n+2, \mathbb{C})$ and only if one uses the representation spaces $\tilde{C}$ comprised of $\mathbb{C}$ - valued functions [14], [16]. Naturally, d_1 and d'_1 coincide with d^p and d'^p, resp., (with $p = m_{\tilde h+1} - m_{\tilde h}$), whenever they act on ERs which are induced from symmetric traceless tensors of $M = SO(n)$. The multiplets are shown in Fig. 1, Fig. 2, for n even, odd, resp.; the double arrows are the integral intertwining operators (1.27), and the other arrows are the intertwining differential operators (1.28).

For the group $SU^*(4) \cong Spin(5,1)$ the integral operators and the four different differential operators were given explicitly in [30]. Again the intertwining differential operators are given by powers of four different basic operators. Using Weyl's unitary trick these multiplets may be turned into multiplets describing ERs of the group $SU(2,2)$, cf. [51]. These multiplets may be obtained also from direct $SU(2,2)$ considerations as submultiplets of the maximal (24-member) multiplets, cf. [15], and they will be used here in Section 6 in the q-deformed case.

1.4. *Organization of the rest of the lectures.* Sections 2. and partly 3. review part of the exposition of [23] for $U_q(sl(n))$ with general n. Then in Sections 4. and 5. we consider in detail the case $n = 4$ following part of the exposition of [24]. In Section 6, following mostly [25] and [26], we use q - conformal invariance to propose new q - Minkowski space-time and q - Maxwell hierarchies of equations.

2. The matrix quantum group $\mathrm{GL}_q(\mathrm{n})$ and the dual quantum algebra

In the beginning of this Section we follow [45] and [18]. Let us consider an $n \times n$ matrix M with non-commuting matrix elements a_{ij}, $1 \leq i,j \leq n$, called also *quantum matrix* [45]. Let us denote by $A_q(n)$, $q \in \mathbb{C}$, the bialgebra with unit element $1_{\mathcal{A}}$ and generated by the matrix elements a_{ij} with the following commutation relations [45] $(\lambda = q - q^{-1})$:

$$a_{i\ell}a_{ij} = qa_{ij}a_{i\ell} , \quad \ell > j \tag{2.1a}$$

$$a_{kj}a_{ij} = qa_{ij}a_{kj} , \quad k > i \tag{2.1b}$$

$$a_{kj}a_{i\ell} = a_{i\ell}a_{kj} , \quad k > i , \ \ell > j \tag{2.1c}$$

$$a_{ij}a_{k\ell} = a_{k\ell}a_{ij} - \lambda a_{i\ell}a_{kj}, \quad k > i, \ \ell > j \tag{2.1d}$$

and the following comultiplication $\delta_{\mathcal{A}}$ and counit $\varepsilon_{\mathcal{A}}$:

$$\delta_{\mathcal{A}}(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj} , \quad \varepsilon_{\mathcal{A}}(a_{ij}) = \delta_{ij} \tag{2.2}$$

This bialgebra has an element D called quantum determinant and given by:

$$D = \sum_{\rho \in S_n} \epsilon(\rho)\, a_{1,\rho(1)} \cdots a_{n,\rho(n)} = \sum_{\rho \in S_n} \epsilon(\rho)\, a_{\rho(1),1} \cdots a_{\rho(n),n} \tag{2.3}$$

where summations are over all permutations ρ of $\{1, \ldots, n\}$ and the quantum signature is:

$$\epsilon(\rho) = \prod_{\substack{j<k \\ \rho(j)>\rho(k)}} (-q^{-1}) \tag{2.4}$$

The determinant obeys:

$$\delta_{\mathcal{A}}(D) \;=\; D \otimes D \;, \quad \varepsilon_{\mathcal{A}}(D) \;=\; 1 \tag{2.5}$$

The determinant is central, i.e., it commutes with the elements a_{ik} :

$$a_{ik}\; D \;=\; D\; a_{ik} \tag{2.6}$$

Further, it is assumed that $D \neq 0$ and one considers an extension of this bialgebra by an element D^{-1} which obeys:

$$DD^{-1} \;=\; D^{-1}D \;=\; 1_{\mathcal{A}} \tag{2.7}$$

where $1_{\mathcal{A}}$ denotes the unit element also in the extension. This extension is called the *matrix quantum group* $GL_q(n)$. It is a Hopf algebra with antipode defined as follows. We first define the left and right quantum cofactor matrix A_{ij} :

$$\begin{aligned} A_{ij} \;&=\; \sum_{\rho(i)=j} \frac{\epsilon(\rho\circ\sigma_i)}{\epsilon(\sigma_i)}\; a_{1,\rho(1)} \ldots \widehat{a}_{ij} \ldots a_{n,\rho(n)} \;= \\ &=\; \sum_{\rho(j)=i} \frac{\epsilon(\rho\circ\sigma'_j)}{\epsilon(\sigma'_j)}\; a_{\rho(1),1} \ldots \widehat{a}_{ij} \ldots a_{\rho(n),n} \quad . \end{aligned} \tag{2.8}$$

where σ_i and σ'_j denote the cyclic permutations:

$$\sigma_i \;=\; \{i,\ldots,1\} \;, \quad \sigma'_j \;=\; \{j,\ldots,n\} \tag{2.9}$$

and the notation $\hat{x}$ indicates that x is to be omitted. Now one can show that:

$$\sum_j a_{ij}\; A_{\ell j} \;=\; \sum_j A_{ji}\; a_{j\ell} \;=\; \delta_{i\ell}\; D \tag{2.10}$$

and obtain the left and right inverse:

$$M^{-1} \;=\; D^{-1}\; A \;=\; A\; D^{-1} \tag{2.11}$$

Finally, we can introduce the antipode in $GL_q(n)$:

$$\gamma_{\mathcal{A}}(a_{ij}) \;=\; D^{-1}\; A_{ji} \;=\; A_{ji}\; D^{-1} \tag{2.12}$$

Until here we followed [45] and [18]. Further we follow [23].

We introduce a basis of $GL_q(n)$ which consists of monomials

$$\begin{aligned} f \;&=\; (a_{21})^{m_{21}} \ldots (a_{n,n-1})^{m_{n,n-1}} (a_{11})^{\ell_1} \ldots (a_{nn})^{\ell_n}\; (a_{n-1,n})^{n_{n-1,n}} \ldots (a_{12})^{n_{12}} \;= \\ &=\; f_{\bar{m},\bar{\ell},\bar{n}} \end{aligned} \tag{2.13}$$

where $\bar{m},\bar{\ell},\bar{n}$ denote the sets $\{m_{ij}\}$, $\{\ell_i\}$, $\{n_{ij}\}$, resp., $m_{ij},\ell_i,n_{ij} \in \mathbb{Z}_+$ and we have used the so-called normal ordering of the elements a_{ij}. Namely, we first put the elements a_{ij} with $i > j$ in lexicographic order, i.e., if $i < k$ then a_{ij} $(i > j)$ is before $a_{k\ell}$ $(k > \ell)$ and a_{ti} $(t > i)$ is before a_{tk} $(t > k)$; then we put the elements a_{ii}; finally we put the elements a_{ij} with $i < j$ in antilexicographic order, i.e., if $i > k$ then a_{ij} $(i < j)$ is before $a_{k\ell}$ $(k < \ell)$ and a_{ti} $(t < i)$ is before a_{tk} $(t < k)$. Note also that :

$$f_{\bar{0},\bar{0},\bar{0}} = 1_{\mathcal{A}} \tag{2.14}$$

We need the dual algebra of $GL_q(n)$. This is the algebra $\mathcal{U}_g = U_q(sl(n)) \otimes U_q(\mathcal{Z})$, where $U_q(\mathcal{Z})$ is central in $\mathcal{U}_g$ [21], [29]. Let us denote the Chevalley generators of $sl(n)$ by H_i, $X_i^{\pm}$, $i = 1, \ldots, n-1$. Then we take for the 'Chevalley' generators of $\mathcal{U} = U_q(sl(n))$: $k_i = q^{H_i/2}$, $k_i^{-1} = q^{-H_i/2}$, $X_i^{\pm}$, $i = 1, \ldots, n-1$, with the following *algebra* relations:

$$k_i k_j = k_j k_i \ , \quad k_i k_i^{-1} = k_i^{-1} k_i = 1_{\mathcal{U}_g} \ , \quad k_i X_j^{\pm} = q^{\pm c_{ij}} X_j^{\pm} k_i \tag{2.15a}$$

$$[X_i^+, X_j^-] = \delta_{ij} \left(k_i^2 - k_i^{-2}\right) / \lambda \ , \tag{2.15b}$$

$$\left(X_i^{\pm}\right)^2 X_j^{\pm} - [2]_q X_i^{\pm} X_j^{\pm} X_i^{\pm} + X_j^{\pm} \left(X_i^{\pm}\right)^2 = 0 \ , \quad |i-j| = 1 \ , \tag{2.15c}$$

$$[X_i^{\pm}, X_j^{\pm}] = 0 \ , \quad |i-j| \neq 1 \ , \tag{2.15d}$$

where c_{ij} is the Cartan matrix of $sl(n)$, and *coalgebra* relations :

$$\begin{aligned} \delta_{\mathcal{U}}(k_i^{\pm}) &= k_i^{\pm} \otimes k_i^{\pm} \\ \delta_{\mathcal{U}}(X_i^{\pm}) &= X_i^{\pm} \otimes k_i + k_i^{-1} \otimes X_i^{\pm} \\ \varepsilon_{\mathcal{U}}(k_i^{\pm}) &= 1 \ , \quad \varepsilon_{\mathcal{U}}(X_i^{\pm}) = 0 \\ \gamma_{\mathcal{U}}(k_i) &= k_i^{-1} \ , \quad \gamma_{\mathcal{U}}(X_i^{\pm}) = -q^{\pm 1} X_i^{\pm} \end{aligned} \tag{2.16}$$

where $k_i^+ = k_i$, $k_i^- = k_i^{-1}$. Further, we denote the generator of $\mathcal{Z}$ by H and the generators of $U_q(\mathcal{Z})$ by $k = q^{H/2}$, $k^{-1} = q^{-H/2}$, $kk^{-1} = k^{-1}k = 1_{\mathcal{U}_g}$. The generators k, k^{-1} commute with the generators of $\mathcal{U}$, and their coalgebra relations are as those of any k_i. From now on we shall give most formulae only for the generators k_i, $X_i^{\pm}$, k, since the analogous formulae for k_i^{-1}, k^{-1} follow trivially from those for k_i, k, resp.

The bilinear form giving the duality between $\mathcal{U}_g$ and $GL_q(n)$ is given by [23]:

$$\langle\, k_i \ , \ a_{j\ell} \,\rangle = \delta_{j\ell}\, q^{(\delta_{ij} - \delta_{i,j+1})/2} \tag{2.17a}$$

$$\langle\, X_i^+ \ , \ a_{j\ell} \,\rangle = \delta_{j+1,\ell}\delta_{ij} \tag{2.17b}$$

$$\langle\, X_i^- \ , \ a_{j\ell} \,\rangle = \delta_{j-1,\ell}\delta_{i\ell} \tag{2.17c}$$

$$\langle\, k \ , \ a_{j\ell} \,\rangle = \delta_{j\ell}\, q^{1/2} \tag{2.17d}$$

The pairing between arbitrary elements of $\mathcal{U}_g$ and f follows then from the properties of the duality pairing. The pairing (2.17) is standardly supplemented with

$$\langle\, y \ , \ 1_{\mathcal{A}} \,\rangle = \varepsilon_{\mathcal{U}_g}(y) \tag{2.18}$$

It is well known that the pairing provides the fundamental representation of $\mathcal{U}_g$:

$$F(y)_{j\ell} = \langle y , a_{j\ell} \rangle , \quad y = k_i, X_i^{\pm}, k \tag{2.19}$$

Of course, $F(k) = q^{1/2} I_n$, where I_n is the unit $n \times n$ matrix.

3. Representations of $\mathcal{U}_g$ and $\mathcal{U}$

This Section follows mostly [23]. We begin by defining *two actions* of the dual algebra $\mathcal{U}_g$ on the basis (2.13) of $GL_q(n)$.

First we introduce the *left regular representation* of $\mathcal{U}_g$ for which in the $q = 1$ case we need the infinitesimal version of :

$$\pi(Y)\, M = Y^{-1}\, M , \quad Y, M \in GL(n) \tag{3.1}$$

Explicitly, we define the action of $\mathcal{U}_g$ on $GL_q(n)$ as follows (cf. also (1.6)):

$$\pi(y)\, a_{i\ell} \doteq \left(F\left(\gamma^0_{\mathcal{U}}(y)\right) M\right)_{i\ell} = \sum_j F\left(\gamma^0_{\mathcal{U}}(y)\right)_{ij} a_{j\ell} = \sum_j \langle \gamma^0_{\mathcal{U}}(y) , a_{ij} \rangle\, a_{j\ell} \tag{3.2}$$

where y denotes the generators of $\mathcal{U}_g$ and $\gamma^0_{\mathcal{U}}(y)$ is the antipode action for $q = 1$. From (3.2) we find the explicit action of the generators of $\mathcal{U}_g$:

$$\pi(k_i)\, a_{j\ell} = q^{(\delta_{i+1,j} - \delta_{ij})/2}\, a_{j\ell} \tag{3.3a}$$

$$\pi(X_i^+)\, a_{j\ell} = -\delta_{ij}\, a_{j+1\ell} \tag{3.3b}$$

$$\pi(X_i^-)\, a_{j\ell} = -\delta_{i+1,j}\, a_{j-1\ell} , \tag{3.3c}$$

$$\pi(k)\, a_{j\ell} = q^{-1/2}\, a_{j\ell} \tag{3.3d}$$

The above is supplemented with the following action on the unit element of $GL_q(n)$:

$$\pi(k_i)\, 1_{\mathcal{A}} = 1_{\mathcal{A}} , \quad \pi(X_i^{\pm})\, 1_{\mathcal{A}} = 0 , \quad \pi(k)\, 1_{\mathcal{A}} = 1_{\mathcal{A}} . \tag{3.4}$$

In order to derive the action of $\pi(y)$ on arbitrary elements of the basis (2.13), we use the twisted derivation rule consistent with the coproduct and the representation structure, namely, we take: $\pi(y)\varphi\psi = \pi(\delta'_{\mathcal{U}_g}(y))(\varphi \otimes \psi)$, where $\delta'_{\mathcal{U}_g} = \sigma \circ \delta_{\mathcal{U}_g}$ is the opposite coproduct, (σ is the permutation operator). Thus, we have:

$$\pi(k_i)\varphi\psi = \pi(k_i)\varphi \cdot \pi(k_i)\psi \tag{3.5a}$$

$$\pi(X_i^{\pm})\varphi\psi = \pi(X_i^{\pm})\varphi \cdot \pi(k_i^{-1})\psi + \pi(k_i)\varphi \cdot \pi(X_i^{\pm})\psi \tag{3.5b}$$

$$\pi(k)\varphi\psi = \pi(k)\varphi \cdot \pi(k)\psi \tag{3.5c}$$

From now on we suppose that q is not a nontrivial root of unity. Applying the above rules one obtains:

$$\pi(k_i)(a_{j\ell})^n = q^{n(\delta_{i+1,j} - \delta_{ij})/2}(a_{j\ell})^n \tag{3.6a}$$

$$\pi(X_i^+)(a_{j\ell})^n = -\delta_{ij} c_n (a_{j\ell})^{n-1} a_{j+1\ell} \tag{3.6b}$$

$$\pi(X_i^-)(a_{j\ell})^n = -\delta_{i+1,j}c_n a_{j-1\ell}(a_{j\ell})^{n-1} \tag{3.6c}$$

$$\pi(k)(a_{j\ell})^n = q^{-n/2}(a_{j\ell})^n \tag{3.6d}$$

where

$$c_n = q^{(n-1)/2}\,[n]_q\ , \quad [n]_q = (q^n - q^{-n})/\lambda \tag{3.7}$$

Analogously, we introduce the *right action* (for $U_q(sl(2))$ see also [46]) for which in the classical case (1.7) one needs the infinitesimal counterpart of :

$$\pi_R(Y)\ M = M\ Y\ , \quad Y, M \in GL(n) \tag{3.8}$$

Thus, we define the right action of $\mathcal{U}_g$ as follows (cf. (1.7)):

$$\pi_R(y)\ a_{i\ell} = (MF(y))_{i\ell} = \sum_j a_{ij}\ F(y)_{j\ell} = \sum_j a_{ij}\ \langle\ y\ ,\ a_{j\ell}\ \rangle \tag{3.9}$$

where y denotes the generators of $\mathcal{U}_g$

From (3.9) we find the explicit right action of the generators of $\mathcal{U}_g$:

$$\pi_R(k_i)\ a_{j\ell} = q^{(\delta_{i\ell}-\delta_{i+1,\ell})/2}\ a_{j\ell} \tag{3.10a}$$

$$\pi_R(X_i^+)\ a_{j\ell} = \delta_{i+1,\ell}\ a_{j,\ell-1} \tag{3.10b}$$

$$\pi_R(X_i^-)\ a_{j\ell} = \delta_{i\ell}\ a_{j,\ell+1} \tag{3.10c}$$

$$\pi_R(k)\ a_{j\ell} = q^{1/2}\ a_{j\ell} \tag{3.10d}$$

supplemented by the right action on the unit element:

$$\pi_R(k_i)\ 1_{\mathcal{A}} = 1_{\mathcal{A}}\ , \quad \pi_R(X_i^\pm)\ 1_{\mathcal{A}} = 0\ , \quad \pi_R(k)\ 1_{\mathcal{A}} = 1_{\mathcal{A}} \tag{3.11}$$

The twisted derivation rule is now given by $\pi_R(y)\varphi\psi = \pi_R(\delta_{\mathcal{U}_g}(y))(\varphi\otimes\psi)$, i.e.,

$$\pi_R(k_i)\varphi\psi = \pi_R(k_i)\varphi\cdot\pi_R(k_i)\psi \tag{3.12a}$$

$$\pi_R(X_i^\pm)\varphi\psi = \pi_R(X_i^\pm)\varphi\cdot\pi_R(k_i)\psi + {} + \pi_R(k_i^{-1})\varphi\cdot\pi_R(X_i^\pm)\psi \tag{3.12b}$$

$$\pi_R(k)\varphi\psi = \pi_R(k)\varphi\cdot\pi_R(k)\psi\ , \tag{3.12c}$$

Using this, we find:

$$\pi_R(k_i)(a_{j\ell})^n = q^{n(\delta_{i\ell}-\delta_{i+1,\ell})/2}(a_{j\ell})^n \tag{3.13a}$$

$$\pi_R(X_i^+)(a_{j\ell})^n = \delta_{i+1,\ell}c_n a_{j,\ell-1}(a_{j\ell})^{n-1} \tag{3.13b}$$

$$\pi_R(X_i^-)(a_{j\ell})^n = \delta_{i\ell}c_n(a_{j\ell})^{n-1}a_{j,\ell+1} \tag{3.13c}$$

$$\pi_R(k)(a_{j\ell})^n = q^{n/2}(a_{j\ell})^n \tag{3.13d}$$

Let us now introduce the elements φ as formal power series of the basis (2.13):

$$\varphi = \sum_{\bar{m},\bar{\ell},\bar{n}\in\mathbb{Z}_+} \mu_{\bar{m},\bar{\ell},\bar{n}}\, f_{\bar{m},\bar{\ell},\bar{n}} \tag{3.14}$$

By (3.6) and (3.13) we have defined left and right action of $\mathcal{U}_g$ on φ. As in the classical case the left and right actions commute, and as in [16] we shall use the right covariance to reduce the left regular representation. In particular, we require the right action to mimic properties of a highest weight module, i.e., annihilation by the raising generators X_i^+ and scalar action by the (exponents of the) Cartan operators k_i, k. However, first we have to make a change of basis using the q-analogue of the classical Gauss decomposition. For this we have to suppose that the principal minor determinants of M :

$$\begin{aligned} D_m &= \sum_{\rho\in S_m} \epsilon(\rho)\, a_{1,\rho(1)}\cdots a_{m,\rho(m)} = \\ &= \sum_{\rho\in S_m} \epsilon(\rho)\, a_{\rho(1),1}\cdots a_{\rho(m),m}\ , \quad m\le n \end{aligned} \tag{3.15}$$

are invertible; note that $D_n = D$, $D_{n-1} = A_{nn}$.

Further, for the ordered sets $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$, let ξ^I_J be the r-minor determinant with respect to rows I and columns J such that

$$\xi^I_J = \sum_{\rho\in S_r} \epsilon(\rho)\, a_{i_{\rho(1)}j_1}\cdots a_{i_{\rho(r)}j_r} \tag{3.16}$$

Note that $\xi^{1\,\cdots\,i}_{1\,\cdots\,i} = D_i$. Then one has [1] $(i,j,\ell = 1,\ldots,n)$:

$$a_{i\ell} = \sum_j B_{ij}Z_{j\ell}\ ,\quad B_{i\ell} = \xi^{1\,\cdots\,\ell-1\,i}_{1\,\cdots\,\ell}\, D^{-1}_{\ell-1}\ ,\quad Z_{i\ell} = D_i^{-1}\xi^{1\,\cdots\,i}_{1\,\cdots\,i-1\,\ell} \tag{3.17}$$

$B_{i\ell} = 0$ for $i<\ell$, $Z_{i\ell} = 0$ for $i>\ell$, (which follows from the obvious extension of (3.16) to the case when I, resp. J, is not ordered). Then Z_{ij}, $i<j$, may be regarded as a q-analogue of local coordinates of the q - deformed flag manifold $B\backslash GL(n)$.

For our purposes we need a refinement of this decomposition :

$$B_{i\ell} = \tilde{Y}_{i\ell}D_{\ell\ell}\ ,\quad \tilde{Y}_{i\ell} = \xi^{1\,\cdots\,\ell-1\,i}_{1\,\cdots\,\ell}\, D^{-1}_{\ell}\ ,\quad D_{\ell\ell} = D_\ell D^{-1}_{\ell-1}\ ,\quad (D_0\equiv 1_{\mathcal{A}}) \tag{3.18}$$

where $\tilde{Y}_{j\ell}$, $j>\ell$, may be regarded as a q-analogue of local coordinates of the q - deformed flag manifold $GL(n)/DZ$.

Clearly, we can replace the basis (2.13) of $GL_q(n)$ with a basis in terms of $\tilde{Y}_{i\ell}$, $i>\ell$, D_ℓ, $Z_{i\ell}$, $i<\ell$. (Note that $\tilde{Y}_{ii} = Z_{ii} = 1_{\mathcal{A}}$.) Thus, we consider formal power series:

$$\tilde{\varphi} = \sum_{\substack{\bar{m},\bar{n}\in\mathbb{Z}_+ \\ \bar{\ell}\in\mathbb{Z}}} \mu'_{\bar{\ell},\bar{m},\bar{n}} (\tilde{Y}_{21})^{m_{21}} \dots (\tilde{Y}_{n,n-1})^{m_{n,n-1}} \times \\ \times (D_1)^{\ell_1} \dots (D_n)^{\ell_n} (Z_{n-1,n})^{n_{n-1,n}} \dots (Z_{12})^{n_{12}} \tag{3.19}$$

Now, let us impose right covariance (cf. [16] and (1.10b)) with respect to X_i^+, i.e., we require:

$$\pi_R(X_i^+)\, \tilde{\varphi} = 0 \tag{3.20}$$

First we notice that:

$$\pi_R(X_i^+)\, \xi_J^I = 0 \ , \quad \text{for } J = \{1,\dots,j\} \ , \ \forall\, I \tag{3.21}$$

from which follow:

$$\pi_R(X_i^+)\, D_j = 0 \ , \quad \pi_R(X_i^+)\, \tilde{Y}_{j\ell} = 0 \tag{3.22}$$

On the other hand $\pi_R(X_i^+)$ acts nontrivially on $Z_{j\ell}$:

$$\pi_R(X_i^+)\, Z_{j\ell} = \delta_{i+1,\ell}\, q^{\delta_{ij}/2}\, Z_{j,\ell-1} \tag{3.23}$$

Thus, (3.20) simply means that our functions $\tilde{\varphi}$ do not depend on $Z_{j\ell}$. Thus, the functions obeying (3.20) are:

$$\tilde{\varphi} = \sum_{\bar{\ell}\in\mathbb{Z}\ ,\ \bar{m}\in\mathbb{Z}_+} \mu_{\bar{\ell},\bar{m}} (\tilde{Y}_{21})^{m_{21}} \dots (\tilde{Y}_{n,n-1})^{m_{n,n-1}} (D_1)^{\ell_1} \dots (D_n)^{\ell_n} \tag{3.24}$$

Next, we impose right covariance with respect to k_i, k :

$$\pi_R(k_i)\, \tilde{\varphi} = q^{r_i/2}\, \tilde{\varphi} \tag{3.25a}$$

$$\pi_R(k)\, \tilde{\varphi} = q^{\hat{r}/2}\, \tilde{\varphi} \tag{3.25b}$$

where $r_i, \hat{r}$ are parameters to be specified below. On the other hand using (3.12a,c), (3.13a,c) we have:

$$\pi_R(k_i)\, \xi_J^I = q^{\delta_{ij}/2}\, \xi_J^I \ , \quad \pi_R(k)\, \xi_J^I = q^{j/2}\, \xi_J^I \ , \quad \text{for } J = \{1,\dots,j\} \ , \ \forall\, I \ , \tag{3.26}$$

from which follows:

$$\pi_R(k_i)\, D_j = q^{\delta_{ij}/2}\, D_j \ , \quad \pi_R(k)\, D_j = q^{j/2}\, D_j \ , \tag{3.27a}$$

$$\pi_R(k_i)\, \tilde{Y}_{j\ell} = \tilde{Y}_{j\ell} \ , \quad \pi_R(k)\, \tilde{Y}_{j\ell} = \tilde{Y}_{j\ell} \ , \tag{3.27b}$$

and thus we have:

$$\pi_R(k_i)\, \tilde{\varphi} = q^{\ell_i/2}\, \tilde{\varphi} \tag{3.28a}$$

$$\pi_R(k)\, \tilde{\varphi} = q^{\sum_{j=1}^n j\ell_j/2}\, \tilde{\varphi} \tag{3.28b}$$

Comparing right covariance conditions (3.25) with the direct calculations (3.28) we obtain $\ell_i = r_i$, for $i < n$, $\sum_{j=1}^{n} j\ell_j = \hat{r}$. This means that $r_i, \hat{r} \in \mathbb{Z}$ and that there is no summation in ℓ_i, also $\ell_n = (\hat{r} - \sum_{i=1}^{n-1} ir_i)/n$. Thus, the reduced functions obeying (3.20) and (3.25) are:

$$\tilde{\varphi} = \sum_{\bar{m}\in\mathbb{Z}_+} \mu_{\bar{m}} \, (\tilde{Y}_{21})^{m_{21}} \dots (\tilde{Y}_{n,n-1})^{m_{n,n-1}} \, (D_1)^{r_1} \dots (D_{n-1})^{r_{n-1}} (D_n)^{\hat{\ell}} \qquad (3.29)$$

where $\hat{\ell} = (\hat{r} - \sum_{i=1}^{n-1} ir_i)/n$.

Next we would like to derive the $\mathcal{U}_g$ - action π on $\tilde{\varphi}$. First, we notice that $\mathcal{U}$ acts trivially on $D_n = D$:

$$\pi(X_i^{\pm}) \, D = 0 \, , \quad \pi(k_i) \, D = D \qquad (3.30)$$

Then we note:

$$\pi(k) \, D_j = q^{-j/2} \, D_j \, , \quad \pi(k) \, \tilde{Y}_{j\ell} = \tilde{Y}_{j\ell} \qquad (3.31)$$

from which follows:

$$\pi(k) \, \tilde{\varphi} = q^{-\hat{r}/2} \, \tilde{\varphi} \qquad (3.32)$$

Thus, the action of $\mathcal{U}$ involves only the parameters r_i, $i < n$, while the action of $U_q(\mathcal{Z})$ involves only the parameter $\hat{r}$. Thus we can consistently also from the representation theory point of view restrict to the matrix quantum group $SL_q(n)$, i.e., we set:

$$D = D^{-1} = 1_{\mathcal{A}} \qquad (3.33)$$

Then the dual algebra is $\mathcal{U} = U_q(sl(n))$. This is justified as in the $q = 1$ case [16] since for our considerations only the semisimple part of the algebra is important. (This would not be possible for the multiparameter deformation of $GL(n)$ [56], [53], since there D is not central. Nevertheless, we expect most of the essential features of our approach to be preserved since the dual algebra can be transformed as a commutation algebra to the one-parameter $\mathcal{U}_g$, with the extra parameters entering only the co-algebra structure [21], [29].)

Thus, the reduced functions for the $\mathcal{U}$ action are:

$$\tilde{\varphi}(\bar{Y}, \bar{D}) = \sum_{\bar{m}\in\mathbb{Z}_+} \mu_{\bar{m}} \, (\tilde{Y}_{21})^{m_{21}} \dots (\tilde{Y}_{n,n-1})^{m_{n,n-1}} \times$$
$$\times \, (D_1)^{r_1} \dots (D_{n-1})^{r_{n-1}} = \qquad (3.34a)$$
$$= \hat{\varphi}(\bar{Y}) \, (D_1)^{r_1} \dots (D_{n-1})^{r_{n-1}} \qquad (3.34b)$$

where $\bar{Y}, \bar{D}$ denote the variables $\tilde{Y}_{il}$, $i > \ell$, D_i, $i < n$.

Further we note the commutation relations of the $\tilde{Y}_{ij}$ and D_i variables:

$$\tilde{Y}_{i\ell}\tilde{Y}_{ij} = q\tilde{Y}_{ij}\tilde{Y}_{i\ell} \, , \quad i > \ell > j \qquad (3.35a)$$
$$\tilde{Y}_{kj}\tilde{Y}_{ij} = q\tilde{Y}_{ij}\tilde{Y}_{kj} \, , \quad k > i > j \qquad (3.35b)$$
$$\tilde{Y}_{kj}\tilde{Y}_{i\ell} = \tilde{Y}_{i\ell}\tilde{Y}_{kj} \, , \quad k > i > \ell > j \qquad (3.35c)$$
$$\tilde{Y}_{k\ell}\tilde{Y}_{ij} = \tilde{Y}_{ij}\tilde{Y}_{k\ell} + \lambda\tilde{Y}_{i\ell}\tilde{Y}_{kj} \, , \quad k > i, \ell > j \, , \quad i \neq \ell \qquad (3.35d)$$

$$\tilde{Y}_{ki}\tilde{Y}_{ij} = q^{-1}\tilde{Y}_{ij}\tilde{Y}_{ki} + q^{-1}\lambda\tilde{Y}_{kj} , \quad k > i > j \tag{3.35e}$$

$$Y_{j\ell}D_i = D_iY_{j\ell} , \quad j > \ell > i \tag{3.36a}$$

$$Y_{j\ell}D_i = qD_iY_{j\ell} , \quad j > i \geq \ell \tag{3.36b}$$

$$Y_{j\ell}D_i = D_iY_{j\ell} , \quad i \geq j > \ell \tag{3.36c}$$

where in $(3.35d)$ we use $\tilde{Y}_{i\ell} = 0$ when $i < \ell$. Note that $(3.35a - d)$ may be obtained by replacing $a_{i\ell}$ with $\tilde{Y}_{i\ell}$ in $(2.1a - d)$. Note that the structure of the q - deformed flag manifold for general n is exhibited already for $n = 4$, while for $n = 3$ relations $(3.35c, d)$ are not present. The commutation relations between the Z and D variables are obtained from (3.35), (3.36), by just replacing Y_{st} by Z_{ts} in all formulae.

Next we obtain:

$$\pi(k_i)\, D_j = q^{-\delta_{ij}/2}\, D_j \tag{3.37a}$$

$$\pi(X_i^+)\, D_j = -\delta_{ij}\, \tilde{Y}_{j+1,j}\, D_j \tag{3.37b}$$

$$\pi(X_i^-)\, D_j = 0 \tag{3.37c}$$

$$\pi(k_i)\, \tilde{Y}_{j\ell} = q^{\frac{1}{2}(\delta_{i+1,j}-\delta_{ij}-\delta_{i+1,\ell}+\delta_{i\ell})}\, \tilde{Y}_{j\ell} \tag{3.38a}$$

$$\begin{aligned}\pi(X_i^+)\, \tilde{Y}_{j\ell} = &-\delta_{ij}\, \tilde{Y}_{j+1,\ell} + \delta_{i\ell}\, q^{1-\delta_{j,\ell+1}/2}\, \tilde{Y}_{\ell+1,\ell}\, \tilde{Y}_{j\ell} + \\ &+ \delta_{i+1,\ell}\left(q^{-1}\, \tilde{Y}_{j,\ell-1} - \tilde{Y}_{\ell,\ell-1}\, \tilde{Y}_{j\ell}\right)\end{aligned} \tag{3.38b}$$

$$\pi(X_i^-)\, \tilde{Y}_{j\ell} = -\delta_{i+1,j}\, q^{-\delta_{i\ell}/2}\, \tilde{Y}_{j-1,\ell} \tag{3.38c}$$

These results have the important consequence that the degrees of the variables D_j are not changed by the action of $\mathcal{U}$. Thus, the parameters r_i indeed characterize the action of $\mathcal{U}$, i.e., we have obtained representations of $\mathcal{U}$, and it is easy to check that π satisfy (2.15). To obtain the representations more explicitly one just applies the above formulae to our basis using the twisted derivation rule (3.5). In particular, we have:

$$\pi(k_i)\,(D_j)^n = q^{-n\delta_{ij}/2}\,(D_j)^n , \quad n \in \mathbb{Z} , \tag{3.39a}$$

$$\pi(X_i^+)\,(D_j)^n = -\delta_{ij}\,\bar{c}_n\, \tilde{Y}_{j+1,j}\,(D_j)^n , \quad n \in \mathbb{Z} , \tag{3.39b}$$

$$\pi(X_i^-)\,(D_j)^n = 0 , \quad n \in \mathbb{Z} , \tag{3.39c}$$

$$\pi(k_i)\,(\tilde{Y}_{j\ell})^n = q^{\frac{n}{2}(\delta_{i+1,j}-\delta_{ij}-\delta_{i+1,\ell}+\delta_{i\ell})}\,(\tilde{Y}_{j\ell})^n , \quad n \in \mathbb{Z}_+ , \tag{3.40a}$$

$$\begin{aligned}\pi(X_i^+)\,(\tilde{Y}_{j\ell})^n = &-\delta_{ij}\, c_n\,(\tilde{Y}_{j\ell})^{n-1}\,\tilde{Y}_{j+1,\ell} + \\ &+ \delta_{i+1,\ell}\,\bar{c}_n\left(q^{-1}\,\tilde{Y}_{j,\ell-1}\,(\tilde{Y}_{j\ell})^{n-1} - \tilde{Y}_{\ell,\ell-1}\,(\tilde{Y}_{j\ell})^n\right) + \\ &+ \delta_{i\ell}\, q^{1-n\delta_{j,\ell+1}/2}\, c_n\, \tilde{Y}_{\ell+1,\ell}\,(\tilde{Y}_{j\ell})^n , \quad n \in \mathbb{Z}_+\end{aligned} \tag{3.40b}$$

$$\pi(X_i^-)\,(\tilde{Y}_{j\ell})^n = -\delta_{i+1,j}\, q^{-\delta_{j,\ell+1}n/2}\, c_n\, \tilde{Y}_{j-1,\ell}\,(\tilde{Y}_{j\ell})^{n-1} , \quad n \in \mathbb{Z}_+ , \tag{3.40c}$$

where

$$\bar{c}_n = q^{(1-n)/2}\,[n]_q \tag{3.41}$$

It is easy to check that $\pi_{\bar r}$ satisfy (2.15).

We shall denote by $\mathcal{C}_{\bar r}$ the representation space of functions in (3.34) which have covariance properties (3.20), (3.25a). The representation acting in $\mathcal{C}_{\bar r}$ we denote by $\tilde\pi_{\bar r}$ doing also a renormalization to simplify things later, namely, we set:

$$\tilde\pi_{\bar r}(k_i) \ = \ \pi(k_i) \ , \qquad \tilde\pi_{\bar r}(X_i^\pm) \ = \ q^{\pm(r_i-1)/2}\, \pi(X_i^\pm) \tag{3.42}$$

Then $\tilde\pi_{\bar r}$ also satisfy (2.15).

Further, since the action of $\mathcal{U}$ is not affecting the degrees of D_i, we introduce (as in [16]) the restricted functions $\hat\varphi(\bar Y)$ by the formula which is prompted in (3.34b) :

$$\hat\varphi(\bar Y) \ \equiv \ (\mathcal{A}\tilde\varphi)(\bar Y) \ \doteq \ \tilde\varphi(\bar Y, D_1 = \cdots = D_{n-1} = 1_{\mathcal{A}}) \tag{3.43a}$$

$$\hat\varphi(\bar Y) \ = \ \sum_{\bar m\in \mathbb{Z}_+} \mu_{\bar m}\, (\tilde Y_{21})^{m_{21}} \dots (\tilde Y_{n,n-1})^{m_{n,n-1}} \tag{3.43b}$$

We denote the representation space of $\hat\varphi(\bar Y)$ by $\hat{\mathcal{C}}_{\bar r}$ and the representation acting in $\hat{\mathcal{C}}_{\bar r}$ by $\hat\pi_{\bar r}$. Thus, the operator $\mathcal{A}$ acts from $\mathcal{C}_{\bar r}$ to $\hat{\mathcal{C}}_{\bar r}$. The properties of $\hat{\mathcal{C}}_{\bar r}$ follow from the intertwining requirement for $\mathcal{A}$ [16]:

$$\hat\pi_{\bar r} \circ \mathcal{A} \ = \ \mathcal{A} \circ \tilde\pi_{\bar r} \tag{3.44}$$

We have defined the representations $\hat\pi_{\bar r}$ for $r_i \in \mathbb{Z}$. However, notice that we can consider the restricted functions $\hat\varphi(\bar Y)$ for arbitrary complex r_i. We shall make these extension from now on, since this gives the same set of representations for $U_q(sl(n))$ as in the case $q = 1$.

For the more compact exposition of the representation formulae we shall need below also the following operators (corresponding to each $\tilde Y_{j\ell}$) :

$$\hat M_{j\ell}\, \hat\varphi(\bar Y) \ = \ \sum_{\bar m\in \mathbb{Z}_+} \mu_{\bar m}\, \hat M_{j\ell}\, (\tilde Y_{21})^{m_{21}} \dots (\tilde Y_{n,n-1})^{m_{n,n-1}} \tag{3.45a}$$

$$T_{j\ell}\, \hat\varphi(\bar Y) \ = \ \sum_{\bar m\in \mathbb{Z}_+} \mu_{\bar m}\, T_{j\ell}\, (\tilde Y_{21})^{m_{21}} \dots (\tilde Y_{n,n-1})^{m_{n,n-1}} \tag{3.45b}$$

$$\hat M_{j\ell}\, \tilde f_{\bar m} \ = (\tilde Y_{21})^{m_{21}} \dots (\tilde Y_{j\ell})^{m_{j\ell}+1} \dots (\tilde Y_{n,n-1})^{m_{n,n-1}} \tag{3.46a}$$

$$T_{j\ell}\, \tilde f_{\bar m} \ = q^{m_{j\ell}}\, \tilde f_{\bar m} \tag{3.46b}$$

$$\tilde f_{\bar m} \ = (\tilde Y_{21})^{m_{21}} \dots (\tilde Y_{n,n-1})^{m_{n,n-1}} \tag{3.46c}$$

Using this we define the q-difference operators by:

$$\hat{\mathcal{D}}_{j\ell}\, \hat\varphi(\bar Y) \ = \ \frac{1}{\lambda}\, \hat M_{j\ell}^{-1} \left(T_{j\ell} - T_{j\ell}^{-1} \right) \hat\varphi(\bar Y) \tag{3.47}$$

from which follows:

$$\hat{\mathcal{D}}_{j\ell}\, \tilde f_{\bar m} \ = \ [m_{j\ell}]_q\, (\tilde Y_{21})^{m_{21}} \dots (\tilde Y_{j\ell})^{m_{j\ell}-1} \dots (\tilde Y_{n,n-1})^{m_{n,n-1}} \tag{3.48}$$

Of course, for $q \to 1$ we have $\hat{\mathcal{D}}_{j\ell} \to \partial_{Y_{j\ell}} \equiv \partial/\partial Y_{j\ell}$. (Note that the above operators for different variables commute, i.e., with these we have actually passed to commuting variables.)

For the intertwining operators between partially equivalent representations we need the action of $\pi_R(X_i^-)$ on $\tilde{Y}_{j\ell}$ and D_ℓ. Using (3.10) and (3.12) we obtain:

$$\pi_R(X_i^-)(D_\ell)^n = \delta_{i\ell} c_n \, (D_\ell)^n Z_{\ell,\ell+1} \tag{3.49a}$$

$$\pi_R(X_i^-)\, (\tilde{Y}_{j\ell})^n = \delta_{i\ell}\, q^{n-3/2}\, [n]_q\, (\tilde{Y}_{j\ell})^{n-1}\, \tilde{Y}_{j,\ell+1}\, D_{\ell+1}\, D_\ell^{-2}\, D_{\ell-1} \tag{3.49b}$$

where, as usual, we use $\tilde{Y}_{jj} = 1_{\mathcal{A}} = D_0$. We shall use also the repeated action of $\pi_R(X_i^-)$ so in addition we need:

$$\pi_R(X_i^-)\, Z_{j\ell} = \delta_{i\ell}\, Z_{j,\ell+1} - \delta_{ij}\, q^{-\delta_{j+1,\ell}/2}\, Z_{j,j+1}\, Z_{j\ell} + \delta_{i,j-1}\, D_j^{-1}\, \xi^{1\,\cdots\, j}_{1\,\cdots\, j-2,j,\ell} \tag{3.50}$$

$$\pi_R(k_i)\, Z_{j\ell} = q^{(\delta_{i+1,j} - \delta_{ij} + \delta_{i\ell} - \delta_{i+1,\ell})/2}\, Z_{j\ell} \tag{3.51}$$

4. The case of $U_q(sl(4))$

In this Section we consider in more detail the case $n = 4$, following mostly [24]. [For $n = 2, 3$, resp., we refer to [22], [23], resp.]

It is convenient (also for the comparison with the $q = 1$ case) to make the following change of variables:

$$\begin{aligned} Y_{31} &= \tilde{Y}_{31} - q\tilde{Y}_{21}\tilde{Y}_{32} , \quad Y_{41} = \tilde{Y}_{41} - q\tilde{Y}_{21}\tilde{Y}_{42} , \\ Y_{21} &= -q\tilde{Y}_{21} , \quad Y_{43} = q\tilde{Y}_{43} , \quad Y_{ij} = \tilde{Y}_{ij} , \quad \text{for } (ij) = (32),(42) . \end{aligned} \tag{4.1}$$

Using (3.35) we have:

$$Y_{i\ell} Y_{ij} = q^{1-2\delta_{\ell 2}} Y_{ij} Y_{i\ell} , \quad 4 \geq i > \ell > j \geq 1 , \tag{4.2a}$$
$$Y_{kj} Y_{ij} = q^{1-2\delta_{i2}} Y_{ij} Y_{kj} , \quad 4 \geq k > i > j \geq 1 , \tag{4.2b}$$
$$Y_{41} Y_{32} = Y_{32} Y_{41} + \lambda Y_{31} Y_{42} , \tag{4.2c}$$
$$Y_{4i} Y_{j1} = Y_{j1} Y_{4i} , \quad (ij) = (23),(32) , \tag{4.2d}$$
$$Y_{ki} Y_{ij} = q^{1-2\delta_{i3}} Y_{ij} Y_{ki} - (-1)^{\delta_{i3}} \lambda Y_{kj} , \quad 4 \geq k > i > j \geq 1 , \tag{4.2e}$$

(each of (4.2a, b, e) has four cases). Note that (3.36) holds also for $Y_{j\ell}$ replacing $\tilde{Y}_{j\ell}$. Note that for q a phase ($|q| = 1$) the q - flag manifold in the Y coordinates is invariant under the anti-linear anti-involution ω acting as:

$$\omega(Y_{j\ell}) = Y_{5-\ell,5-j} . \tag{4.3}$$

Thus it can be considered as a q - deformed flag manifold of the quantum group $SU_q(2,2)$.

The reduced functions for the $\mathcal{U}$ action are (cf. (3.34)):

$$\tilde{\varphi}(\bar{Y},\bar{D}) = \sum_{i,j,k,\ell,m,n\in\mathbb{Z}_+} \mu_{ijk\ell mn}\, \tilde{\varphi}_{ijk\ell mn} \tag{4.4a}$$

$$\begin{aligned}\tilde{\varphi}_{ijk\ell mn} &= (Y_{21})^i\, (Y_{31})^j\, (Y_{32})^k\, (Y_{41})^\ell\, (Y_{42})^m\, (Y_{43})^n \times \\ &\quad \times\, (D_1)^{r_1}\, (D_2)^{r_2}\, (D_3)^{r_3}\end{aligned} \tag{4.4b}$$

Now the action of $U_q(sl(4))$ on (4.4) is given explicitly by:

$$\hat{\pi}_{\bar{r}}(k_1)\, \tilde{\varphi}_{ijk\ell mn} = q^{i+(j-k+\ell-m-r_1)/2}\, \tilde{\varphi}_{ijk\ell mn}\ , \tag{4.5a}$$

$$\hat{\pi}_{\bar{r}}(k_2)\, \tilde{\varphi}_{ijk\ell mn} = q^{k+(-i+j+m-n-r_2)/2}\, \tilde{\varphi}_{ijk\ell mn}\ , \tag{4.5b}$$

$$\hat{\pi}_{\bar{r}}(k_3)\, \tilde{\varphi}_{ijk\ell mn} = q^{n+(-j-k+\ell+m-r_3)/2}\, \tilde{\varphi}_{ijk\ell mn}\ , \tag{4.5c}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_1^+)\, \tilde{\varphi}_{ijk\ell mn} &= q^{-1+(-j+k-\ell+m)/2}\, [r_1-i]_q\, \tilde{\varphi}_{i+1,jk\ell mn} + \\ &+\, q^{i-r_1-1+(j-k-\ell+m)/2}\, [k]_q\, \tilde{\varphi}_{i,j+1,k-1,\ell mn} + \\ &+\, q^{i-r_1-1+(j-k+\ell-m)/2}\, [m]_q\, \tilde{\varphi}_{ijk,\ell+1,m-1,n}\ ,\end{aligned} \tag{4.6a}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_2^+)\, \tilde{\varphi}_{ijk\ell mn} &= q^{r_2-k+(i-j-m+n)/2}\, [i]_q\, \tilde{\varphi}_{i-1,j+1,k\ell mn} + \\ &+\, q^{(i+j+m-n)/2}\, [j-i+k+m-n-r_2]_q\, \tilde{\varphi}_{ij,k+1,\ell mn} + \\ &+\, q^{-r_2+(-i+j+k+3m-3n)/2}\, [\ell]_q\, \tilde{\varphi}_{i,j+1,k,\ell-1,m+1,n} + \\ &+\, q^{k-r_2+(-i+j+m-n)/2}\, [n]_q\, \tilde{\varphi}_{ijk\ell,m+1,n-1}\ ,\end{aligned} \tag{4.6b}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_3^+)\, \tilde{\varphi}_{ijk\ell mn} &= -\, q^{r_3-1-n+(j+k-\ell-m)/2}\, [j]_q\, \tilde{\varphi}_{i,j-1,k,\ell+1,mn} - \\ &-\, q^{r_3-1-n+(3j+k-3\ell-m)/2}\, [k]_q\, \tilde{\varphi}_{ij,k-1,\ell,m+1,n} + \\ &+\, q^{-1+(-j-k+\ell+m)/2}\, [n-r_3]_q\, \tilde{\varphi}_{ijk\ell m,n+1}\ ,\end{aligned} \tag{4.6c}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_1^-)\, \tilde{\varphi}_{ijk\ell mn} &= q^{1+(-j+k-\ell+m)/2}\, [i]_q\, \tilde{\varphi}_{i-1,jk\ell mn} + \\ &+\, q^{i+2+(-j+k-\ell+m)/2}\, [j]_q\, \tilde{\varphi}_{i,j-1,k+1,\ell mn} + \\ &+\, q^{i+2+(j-k-\ell+m)/2}\, [\ell]_q\, \tilde{\varphi}_{ijk,\ell-1,m+1,n}\ ,\end{aligned} \tag{4.7a}$$

$$\hat{\pi}_{\bar{r}}(X_2^-)\, \tilde{\varphi}_{ijk\ell mn} = -\, q^{(-i+j-m+n)/2}\, [k]_q\, \tilde{\varphi}_{ij,k-1,\ell mn}\ , \tag{4.7b}$$

$$\hat{\pi}_{\bar{r}}(X_3^-)\ \tilde{\varphi}_{ijk\ell mn} \;=\; -\ q^{-n+(-j-3k+\ell+3m)/2}\ [\ell]_q\ \tilde{\varphi}_{i,j+1,k,\ell-1,mn} - \\ -\ q^{-n+(-j-k+\ell+m)/2}\ [m]_q\ \tilde{\varphi}_{ij,k+1,\ell,m-1,n} - \\ -\ q^{1+(-j-k+\ell+m)/2}\ [n]_q\ \tilde{\varphi}_{ijk\ell m,n-1}\ . \qquad (4.7c)$$

It is easy to check that $\hat{\pi}_{\bar{r}}(k_i)$, $\hat{\pi}_{\bar{r}}(X_i^{\pm})$ satisfy (2.15).

Then we consider the restricted functions (cf. (3.43)):

$$\hat{\varphi}(\bar{Y}) \;=\; \sum_{i,j,k,\ell,m,n\in\mathbb{Z}_+} \mu_{ijk\ell mn}\ \hat{\varphi}_{ijk\ell mn}\ , \qquad (4.8a)$$

$$\hat{\varphi}_{ijk\ell mn} \;=\; (\hat{Y}_{21})^i\ (Y_{31})^j\ (Y_{32})^k\ (Y_{41})^\ell\ (Y_{42})^m\ (Y_{43})^n\ . \qquad (4.8b)$$

As a consequence of the intertwining property (3.44) we obtain that $\hat{\varphi}_{ijk\ell mn}$ obey the same transformation rules (4.5), (4.6), (4.7), as $\tilde{\varphi}_{ijk\ell mn}$.

Recall that we consider the representations $\hat{\pi}_{\bar{r}}$ for arbitrary complex r_i and we know from the general analysis of [23] that whenever some $m_i = r_i + 1$ or $m_{ij} = m_i + \cdots + m_j$, $(i < j)$ is a positive integer the representations are reducible and there exist invariant subspaces. We give now two simple examples.

Let $m_1 = r_1 + 1 \in \mathbb{N}$. Then it is clear that functions $\tilde{\varphi}$ with $\mu_{ijk\ell mn} = 0$ if $i \geq m_1$ form an invariant subspace since:

$$\hat{\pi}_{\bar{r}}(X_1^+)\ \tilde{\varphi}_{r_1,jk\ell mn} \;=\; q^{(j+m-\ell-2-k)/2}\ [k]_q\ \tilde{\varphi}_{r_1,j+1,k-1,\ell mn} + \\ +\ q^{(j+\ell-k-2-m)/2}\ [m]_q\ \tilde{\varphi}_{r_1,jk,\ell+1,m-1,n}\ , \qquad (4.9)$$

and all other operators in (4.5), (4.6), (4.7) either preserve or lower the index i. The same is true for the functions $\hat{\varphi}$. In particular, for $r_1 = 0$ the functions in the invariant subspace do not depend on the variable Y_{21}.

Analogously if $m_3 = r_3 + 1 \in \mathbb{N}$ the functions $\tilde{\varphi}$ with $\mu_{ijk\ell mn} = 0$ if $n \geq m_3$ form an invariant subspace since:

$$\hat{\pi}_{\bar{r}}(X_3^+)\ \tilde{\varphi}_{ijk\ell m,r_3} \;=\; -\ q^{(k+j+m-\ell-2)/2}\ [j]_q\ \tilde{\varphi}_{i,j-1,k,\ell+1,m,r_3} - \\ -\ q^{(k+3j+m-3\ell-2)/2}\ [m]_q\ \tilde{\varphi}_{ij,k-1,\ell,m+1,r_3}\ , \qquad (4.10)$$

and all other operators in (4.5), (4.6), (4.7) either preserve or lower the index n, the same holding for the functions $\hat{\varphi}$. In particular, for $r_3 = 0$ the functions in the invariant subspace do not depend on the variable Y_{43}.

It will be convenient to use also the following notation for the coordinates of the flag manifold:

$$\xi = Y_{21}\ ,\quad x = Y_{31}\ ,\quad u = Y_{32}\ ,\quad w = Y_{41}\ ,\quad y = Y_{42}\ ,\quad \eta = Y_{43}\ . \qquad (4.11)$$

The above notation we shall employ also for the operators (3.45), (3.47). In terms of the latter operators we rewrite the transformation rules (4.5), (4.6), (4.7) for the functions $\hat{\varphi}$ as follows :

$$\hat{\pi}_{\bar{r}}(k_1)\ \hat{\varphi}(\bar{Y}) \;=\; q^{-r_1/2}\ T_\xi\ (T_x T_w)^{1/2}\ (T_u T_y)^{-1/2}\ \hat{\varphi}(\bar{Y})\ , \qquad (4.12a)$$

$$\hat{\pi}_{\bar{r}}(k_2)\,\hat{\varphi}(\bar{Y}) = q^{-r_2/2}\,T_u\,(T_xT_y)^{1/2}\,(T_\xi T_\eta)^{-1/2}\,\hat{\varphi}(\bar{Y})\,, \tag{4.12b}$$

$$\hat{\pi}_{\bar{r}}(k_3)\,\hat{\varphi}(\bar{Y}) = q^{-r_3/2}\,T_\eta\,(T_wT_y)^{1/2}\,(T_xT_u)^{-1/2}\,\hat{\varphi}(\bar{Y})\,, \tag{4.12c}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_1^+)\,\hat{\varphi}(\bar{Y}) = {} & (1/\lambda)\,q^{-1}\,\hat{M}_\xi\,(T_uT_y)^{1/2}\,(T_xT_w)^{-1/2}\left(q^{r_1}T_\xi^{-1} - q^{-r_1}T_\xi\right)\hat{\varphi}(\bar{Y}) + \\ & + q^{-r_1-1}\,\hat{M}_x\,\hat{\mathcal{D}}_u\,T_\xi\,(T_xT_y)^{1/2}\,(T_uT_w)^{-1/2}\,\hat{\varphi}(\bar{Y}) + \\ & + q^{-r_1-1}\,\hat{M}_w\,\hat{\mathcal{D}}_y\,T_\xi\,(T_xT_w)^{1/2}\,(T_uT_y)^{-1/2}\,\hat{\varphi}(\bar{Y})\,,\end{aligned} \tag{4.13a}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_2^+)\,\hat{\varphi}(\bar{Y}) = {} & q^{r_2}\,\hat{M}_x\,\hat{\mathcal{D}}_\xi\,T_u^{-1}\,(T_\xi T_\eta)^{1/2}\,(T_xT_y)^{-1/2}\,\hat{\varphi}(\bar{Y}) + \\ & + (1/\lambda)\,\hat{M}_u\,(T_\xi T_y)^{1/2}\,(T_xT_\eta)^{-1/2}\,\times \\ & \times\left(q^{-r_2}T_xT_uT_y(T_\xi T_\eta)^{-1} - q^{r_2}T_\xi T_\eta(T_xT_uT_y)^{-1}\right)\hat{\varphi}(\bar{Y}) + \\ & + q^{-r_2}\,\hat{M}_x\,\hat{M}_y\,\hat{\mathcal{D}}_w\,(T_x\,T_uT_y^3)^{1/2}\,(T_\xi T_\eta^3)^{-1/2}\,\hat{\varphi}(\bar{Y}) + \\ & + q^{-r_2}\,\hat{M}_y\,\hat{\mathcal{D}}_\eta\,T_u\,(T_xT_y)^{1/2}\,(T_\xi T_\eta)^{-1/2}\,\hat{\varphi}(\bar{Y})\,,\end{aligned} \tag{4.13b}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_3^+)\,\hat{\varphi}(\bar{Y}) = {} & -\,q^{r_3-1}\,\hat{M}_w\,\hat{\mathcal{D}}_x\,T_\eta^{-1}\,(T_xT_u)^{1/2}\,(T_wT_y)^{-1/2}\,\hat{\varphi}(\bar{Y}) - \\ & -\,q^{r_3-1}\,\hat{M}_y\,\hat{\mathcal{D}}_u\,T_\eta^{-1}\,(T_x^3T_u)^{1/2}\,(T_w^3T_y)^{-1/2}\,\hat{\varphi}(\bar{Y}) + \\ & + (1/\lambda)\,q^{-1}\,\hat{M}_\eta\,(T_wT_y)^{1/2}\,(T_xT_u)^{-1/2}\left(q^{-r_3}T_\eta - q^{r_3}T_\eta^{-1}\right)\hat{\varphi}(\bar{Y})\,,\end{aligned} \tag{4.13c}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_1^-)\,\hat{\varphi}(\bar{Y}) = {} & q\,\hat{\mathcal{D}}_\xi\,(T_uT_y)^{1/2}\,(T_xT_w)^{-1/2}\,\hat{\varphi}(\bar{Y}) + \\ & + q^2\,\hat{M}_u\,\hat{\mathcal{D}}_x\,T_\xi\,(T_uT_y)^{1/2}\,(T_xT_w)^{-1/2}\,\hat{\varphi}(\bar{Y}) + \\ & + q^2\,\hat{M}_y\,\hat{\mathcal{D}}_w\,T_\xi\,(T_xT_y)^{1/2}\,(T_uT_w)^{-1/2}\,\hat{\varphi}(\bar{Y})\end{aligned} \tag{4.14a}$$

$$\hat{\pi}_{\bar{r}}(X_2^-)\,\hat{\varphi}(\bar{Y}) = -\,\hat{\mathcal{D}}_u\,(T_xT_\eta)^{1/2}\,(T_\xi T_y)^{-1/2}\,\hat{\varphi}(\bar{Y})\,, \tag{4.14b}$$

$$\begin{aligned}\hat{\pi}_{\bar{r}}(X_3^-)\,\hat{\varphi}(\bar{Y}) = {} & -\,\hat{M}_x\,\hat{\mathcal{D}}_w\,T_\eta^{-1}\,(T_wT_y^3)^{1/2}\,(T_xT_u^3)^{-1/2}\,\hat{\varphi}(\bar{Y}) - \\ & -\,\hat{M}_u\,\hat{\mathcal{D}}_y\,T_\eta^{-1}\,(T_wT_y)^{1/2}\,(T_xT_u)^{-1/2}\,\hat{\varphi}(\bar{Y}) - \\ & -\,q\,\hat{\mathcal{D}}_\eta\,(T_wT_y)^{1/2}\,(T_xT_u)^{-1/2}\,\hat{\varphi}(\bar{Y})\,.\end{aligned} \tag{4.14c}$$

5. Intertwining operators

This Section reviews mostly [24]. The general prescription for finding the intertwining operators is as in the classical case (cf. also [23]). In order to apply this procedure we need the explicit action of $\pi_R(X_i^-)$ on our functions. First we have to calculate the action on the new basis $Y_{j\ell}$. We have instead of (3.49b):

$$\begin{aligned} \pi_R(X_i^-)\,(Y_{j\ell})^n &= (-1)^{\delta_{i1}}\,\delta_{i\ell}\,\delta_{i+1,j}\,q^{n-1/2}\,[n]_q\,\times \\ &\times\,(Y_{i+1,i})^{n-1}\,D_{i+1}\,D_i^{-2}\,D_{i-1}, \quad i=1,3 \\ \pi_R(X_2^-)\,(Y_{j\ell})^n &= (-1)^{\ell}\,q^{(n-2)(\ell-1)+1/2}\,[n]_q\,Y_{2\ell}\,(Y_{j\ell})^{n-1}\,Y_{j3}\,D_2\,D_1^{-2}\,, \end{aligned} \tag{5.1}$$

where we again use $D_4 = D_0 = Y_{jj} = 1_{\mathcal{A}}$, $Y_{j\ell} = 0$ for $j < \ell$.

Using (5.1) and (3.49a) we obtain:

$$\begin{aligned} \pi_R(X_1^-)\,\tilde{\varphi}^{m_1,m_2,m_3}_{ijk\ell mn} &= -\,q^{i-j-k-\ell-m+(m_1-2)/2}\,[i]_q\,\tilde{\varphi}^{m_1-2,m_2+1,m_3}_{i-1,jk\ell mn} + \\ &+\,q^{(m_1-2)/2}\,[m_1-1]_q\,\tilde{\varphi}^{m_1,m_2,m_3}_{ijk\ell mn}\,Z_{12}\,, \end{aligned} \tag{5.2a}$$

$$\begin{aligned} \pi_R(X_2^-)\,\tilde{\varphi}^{m_1,m_2,m_3}_{ijk\ell mn} &= q^{2k+\ell+m-n+(m_2-2)/2}\,[j]_q\,\tilde{\varphi}^{m_1+1,m_2-2,m_3+1}_{i+1,j-1,k\ell mn} + \\ &+\,q^{k+\ell+m-n+(m_2-4)/2}\,[k]_q\,\tilde{\varphi}^{m_1+1,m_2-2,m_3+1}_{ij,k-1,\ell mn} + \\ &+\,q^{k-j+2m-n+(m_2-4)/2}\,[\ell]_q\,\tilde{\varphi}^{m_1+1,m_2-2,m_3+1}_{i+1,jk,\ell-1,m,n+1} + \\ &+\,q^{m-n+(m_2-6)/2}\,[m]_q\,\tilde{\varphi}^{m_1+1,m_2-2,m_3+1}_{ijk\ell,m-1,n+1} - \\ &-\,q^{2m-n+(m_2-4)/2}\,\lambda\,[k]_q\,[\ell]_q\,\tilde{\varphi}^{m_1+1,m_2-2,m_3+1}_{i,j+1,k-1,\ell-1,m,n+1} + \\ &+\,q^{(m_2-2)/2}\,[m_2-1]_q\,\tilde{\varphi}^{m_1,m_2,m_3}_{ijk\ell mn}\,Z_{23}\,, \end{aligned} \tag{5.2b}$$

$$\begin{aligned} \pi_R(X_3^-)\,\tilde{\varphi}^{m_1,m_2,m_3}_{ijk\ell mn} &= q^{n+(m_3-2)/2}\,[n]_q\,\tilde{\varphi}^{m_1,m_2+1,m_3-2}_{ijk\ell m,n-1} + \\ &+\,q^{(m_3-2)/2}\,[m_3-1]_q\,\tilde{\varphi}^{m_1,m_2,m_3}_{ijk\ell mn}\,Z_{34}\,, \end{aligned} \tag{5.2c}$$

where we have labelled the functions also with the representation parameters $m_s = r_s + 1$. As in the classical case [16] the right action is taking out from the representation space $\mathcal{C}_{\bar{r}}$, and while some of the terms are functions from other representation spaces (depending on which X_s^- is acting), there are terms involving the $Z_{j\ell}$ variables which do not belong to any of our representation spaces. The terms with $Z_{j\ell}$ vanish exactly when $m_s \in \mathbb{N}$ and we take $(\pi_R(X_s^-))^{m_s}$ [16], [23]. Indeed, we know from the general prescription (cf. (1.15), [16], [23]) that if $m_s \in \mathbb{N}$ then there exists an intertwining operator $I_s^{m_s} = (\pi_R(X_s^-))^{m_s}$. We have the following intertwining properties:

$$\begin{aligned} I_1^{m_1}\circ\pi_{m_1,m_2,m_3} &= \pi_{-m_1,m_{12},m_3}\circ I_1^{m_1}\,, \quad m_1\in\mathbb{N} \\ I_2^{m_2}\circ\pi_{m_1,m_2,m_3} &= \pi_{m_{12},-m_2,m_{23}}\circ I_2^{m_2}\,, \quad m_2\in\mathbb{N}\,, \\ I_3^{m_3}\circ\pi_{m_1,m_2,m_3} &= \pi_{m_1,m_{23},-m_3}\circ I_3^{m_3}\,, \quad m_3\in\mathbb{N} \end{aligned} \tag{5.3}$$

The explicit expressions for two of these operators are:

$$(\pi_R(X_1^-))^{m_1}\ \tilde{\varphi}^{m_1,m_2,m_3}_{ijk\ell mn} = (-1)^{m_1}\ q^{m_1(i-m_1/2)}\ \frac{[i]_q!}{[i-m_1]_q!}\ \tilde{\varphi}^{-m_1,m_{12},m_3}_{i-m_1,jk\ell mn} \tag{5.4}$$

$$(\pi_R(X_3^-))^{m_3}\ \tilde{\varphi}^{m_1,m_2,m_3}_{ijk\ell mn} = q^{m_3(i-m_3/2)}\ \frac{[n]_q!}{[n-m_3]_q!}\ \tilde{\varphi}^{m_1,m_{23},-m_3}_{ijk\ell m,n-m_3} \tag{5.5}$$

Having in mind the preceding discussion let us introduce the following q-difference operators (using notation (3.45), (3.47), (4.11)):

$$\hat{I}_1 \equiv -\ q^{(m_1-2)/2}\ \hat{\mathcal{D}}_\xi\ T_\xi\ (T_x T_u T_w T_y)^{-1} \tag{5.6a}$$

$$\begin{aligned}\hat{I}_2 \equiv\ & q^{(m_2-4)/2}\Big(q\ \hat{M}_\xi\ \hat{\mathcal{D}}_x\ T_u + \hat{\mathcal{D}}_u + \\ & + \hat{M}_\xi\ \hat{M}_\eta\ \hat{\mathcal{D}}_w\ (T_x T_w)^{-1}\ T_y + q^{-1}\ \hat{M}_\eta\ \hat{\mathcal{D}}_y\ (T_u T_w)^{-1} - \\ & - \lambda\ \hat{M}_x\ \hat{M}_\eta\ \hat{\mathcal{D}}_u\ \hat{\mathcal{D}}_w\ (T_u T_w)^{-1}\ T_y\Big)\ T_u\ T_w\ T_y\ T_\eta^{-1}\end{aligned} \tag{5.6b}$$

$$\hat{I}_3 \equiv q^{(m_3-2)/2}\ \hat{\mathcal{D}}_\eta\ T_\eta \tag{5.6c}$$

It is not difficult to see that if $m_s \in I\!N$ we have:

$$\hat{I}_s^{m_s} = I_s^{m_s} = (\pi_R(X_s^-))^{m_s}\ .. \tag{5.7}$$

Let us consider now the intertwining operators corresponding to the two non-simple non-highest roots α_{12}, α_{23} which are realized when $m_{12} \in I\!N$, $m_{23} \in I\!N$, resp. In these cases the intertwining operators (up to an overall multiplicative constant) are given by :

$$I^m_{ij} = \sum_{k=0}^{m} a_k\ (\pi_R(X_i^-))^{m-k}\ (\pi_R(X_j^-))^m\ (\pi_R(X_i^-))^k\ , \qquad m = m_{ij}\ ,\quad (ij) = (12),(23), \tag{5.8a}$$

$$a_k = (-1)^k\ a\ \frac{[m_i]_q}{[m_i-k]_q}\ \binom{m}{k}_q\ ,\quad k = 0,\dots,m,\quad a \neq 0, \tag{5.8b}$$

or equivalently, by :

$$I^m_{ij} = \sum_{k=0}^{m} a'_k\ (\pi_R(X_j^-))^{m-k}\ (\pi_R(X_i^-))^m\ (\pi_R(X_j^-))^k\ , \qquad m = m_{ij}\ ,\quad (ij) = (12),(23), \tag{5.8c}$$

$$a'_k = (-1)^k\ a'\ \frac{[m_j]_q}{[m_j-k]_q}\ \binom{m}{k}_q\ ,\quad k = 0,\dots,m,\quad a' \neq 0, \tag{5.8d}$$

where we are using the singular vector given in formula (27) of [17].

Let us illustrate the resulting intertwining operators in the cases $m_{12} = 1$,

$m_{23} = 1$. We have (after a suitable renormalization) :

$$I^1_{12}|_{m_{12}=1} \;=\; -[m_1-1]_q\ \pi_R(X_1^-)\ \pi_R(X_2^-)\ +\ [m_1]_q\ \pi_R(X_2^-)\ \pi_R(X_1^-)\ , \tag{5.9a}$$

$$\begin{aligned} I^1_{12}\ \tilde\varphi^{m_1,m_2,m_3}_{ijk\ell mn}|_{m_{12}=1} \;=\; & q^{2i-j-n}\ [m_1-1]_q\ \Big(q^{k+1}\ [j]_q\ \tilde\varphi^{m_1-1,m_2-1,m_3+1}_{i,j-1,k\ell mn}\ + \\ & +\ q^{-j-\ell+m}\ [\ell]_q\ \tilde\varphi^{m_1-1,m_2-1,m_3+1}_{ijk,\ell-1,m,n+1}\Big)\ - \\ & -\ q^{i-j-n-m_1-3}\ [i]_q\ \Big(q^{k+1}\ [j]_q\ \tilde\varphi^{m_1-1,m_2-1,m_3+1}_{i,j-1,k\ell mn}\ + \\ & +\ [k]_q\ \tilde\varphi^{m_1-1,m_2-1,m_3+1}_{i-1,j,k-1,\ell mn}\ + \\ & +\ q^{-j-\ell+m}\ [\ell]_q\ \tilde\varphi^{m_1-1,m_2-1,m_3+1}_{ijk,\ell-1,m,n+1}\ + \\ & +\ q^{-k-\ell-1}\ [m]_q\ \tilde\varphi^{m_1-1,m_2-1,m_3+1}_{i-1,jk\ell,m-1,n+1}\ - \\ & -\ q^{-k-\ell+m}\ \lambda\ [k]_q\ [\ell]_q\ \tilde\varphi^{m_1-1,m_2-1,m_3+1}_{i-1,j+1,k-1,\ell-1,m,n+1}\Big) \end{aligned} \tag{5.9b}$$

$$I^1_{23}|_{m_{23}=1} \;=\; [1-m_3]_q\ \pi_R(X_3^-)\ \pi_R(X_2^-)\ +\ [m_3]_q\ \pi_R(X_2^-)\ \pi_R(X_3^-)\ , \tag{5.10a}$$

$$\begin{aligned} I^1_{12}\ \tilde\varphi^{m_1,m_2,m_3}_{ijk\ell mn}|_{m_{23}=1} \;=\; & -\ q^{k+\ell+m+n-1}\ [m_3-1]_q\ \Big(q^{-k-\ell-1}\ [m]_q\ \tilde\varphi^{m_1+1,m_2-1,m_3-1}_{ijk\ell,m-1,n}\ + \\ & +\ q^{-j-\ell+m}\ [\ell]_q\ \tilde\varphi^{m_1+1,m_2-1,m_3-1}_{i+1,jk,\ell-1,mn}\ - \\ & -\ q^{-k-\ell+m}\ \lambda\ [k]_q\ [\ell]_q\ \tilde\varphi^{m_1+1,m_2-1,m_3-1}_{i,j+1,k-1,\ell-1,mn}\Big)\ - \\ & +\ q^{k+\ell+m+m_3-2}\ [n]_q\ \Big(q^{k+1}\ [j]_q\ \tilde\varphi^{m_1+1,m_2-1,m_3-1}_{i+1,j-1,k\ell m,n-1}\ + \\ & +\ [k]_q\ \tilde\varphi^{m_1+1,m_2-1,m_3-1}_{ij,k-1,\ell m,n-1}\ + \\ & +\ q^{-j-\ell+m}\ [\ell]_q\ \tilde\varphi^{m_1+1,m_2-1,m_3-1}_{i+1,jk,\ell-1,mn}\ + \\ & +\ q^{-k-\ell-1}\ [m]_q\ \tilde\varphi^{m_1+1,m_2-1,m_3-1}_{ijk\ell,m-1,n}\ - \\ & -\ q^{-k-\ell+m}\ \lambda\ [k]_q\ [\ell]_q\ \tilde\varphi^{m_1+1,m_2-1,m_3-1}_{i,j+1,k-1,\ell-1,mn}\Big)\ . \end{aligned} \tag{5.10b}$$

Using the operators $\hat I_s$ the above formulae can be rewritten as:

$$I^1_{12}|_{m_{12}=1} \;=\; [1-m_1]_q\ \hat I_1\ \hat I_2\ +\ [m_1]_q\ \hat I_2\ \hat I_1\ , \tag{5.11a}$$

$$\begin{aligned} I^1_{12}|_{m_{12}=1} \;=\; & [m_1-1]_q\ \Big(q\ \hat{\mathcal D}_x\ T_u\ +\ \hat M_\eta\ \hat{\mathcal D}_w\ (T_xT_w)^{-1}\ T_y\Big)\ T_\xi^2\ (T_xT_\eta)^{-1}\ - \\ & -\ q^{-m_1-1}\ \Big(q\ \hat M_\xi\ \hat{\mathcal D}_x\ T_u\ +\ \hat{\mathcal D}_u\ + \\ & +\ \hat M_\xi\ \hat M_\eta\ \hat{\mathcal D}_w\ (T_xT_w)^{-1}\ T_y\ +\ q^{-1}\ \hat M_\eta\ \hat{\mathcal D}_y\ (T_uT_w)^{-1}\ - \\ & -\ \lambda\ \hat M_x\ \hat M_\eta\ \hat{\mathcal D}_u\ \hat{\mathcal D}_w\ (T_uT_w)^{-1}\ T_y\Big)\ \hat{\mathcal D}_\xi\ T_\xi\ (T_xT_\eta)^{-1} \end{aligned} \tag{5.11b}$$

$$I^1_{23}|_{m_{23}=1} \; = \; [1-m_3]_q \; \hat{I}_3 \; \hat{I}_2 \; + \; [m_3]_q \; \hat{I}_2 \; \hat{I}_3 \; , \tag{5.12a}$$

$$\begin{aligned} I^1_{23}|_{m_{23}=1} \; = \; & - \; q^{-1} \; [m_3-1]_q \; \Big(q^{-1} \; \hat{\mathcal{D}}_y \; (T_u T_w)^{-1} \; + \; \hat{M}_\xi \; \hat{\mathcal{D}}_w \; (T_x T_w)^{-1} \; T_y \; - \\ & - \; \lambda \; \hat{M}_x \; \hat{\mathcal{D}}_u \; \hat{\mathcal{D}}_w \; (T_u T_w)^{-1} \; T_y \; \Big) \; T_u \; T_w \; T_y \; T_\eta \; + \\ & + \; q^{m_3-2} \; \Big(q \; \hat{M}_\xi \; \hat{\mathcal{D}}_x \; T_u \; + \; \hat{\mathcal{D}}_u \; + \\ & + \; \hat{M}_\xi \; \hat{M}_\eta \; \hat{\mathcal{D}}_w \; (T_x T_w)^{-1} \; T_y \; + \; q^{-1} \; \hat{M}_\eta \; \hat{\mathcal{D}}_y \; (T_u T_w)^{-1} \; - \\ & - \; \lambda \; \hat{M}_x \; \hat{M}_\eta \; \hat{\mathcal{D}}_u \; \hat{\mathcal{D}}_w \; (T_u T_w)^{-1} \; T_y \Big) \; \hat{\mathcal{D}}_\eta \; T_u \; T_w \; T_y \; . \end{aligned} \tag{5.12b}$$

6. New q - Minkowski space-time and q - Maxwell equations hierarchy from q - conformal invariance

6.1. The present Section reviews mostly [25] and in the last subsection [26]. We start with the $q=1$ situation and we first write the Maxwell equations in an indexless formulation, trading the indices for two conjugate variables $z, \bar{z}$. This formulation has two advantages. First, it is very simple, and in fact, just with the introduction of an additional parameter, we can describe a whole infinite hierarchy of equations, which we call the *Maxwell hierarchy* . Second, we can easily identify the variables $z, \bar{z}$ and the four Minkowski coordinates with the six local coordinates of a flag manifold of $SU(2,2)$, or of $SL(4)$ with the appropriate conjugation. Thus, one may look at this as a nice example of unifying internal and external degrees of freedom.

Next we give the q - analogs of the above constructions. We recall that the specifics of our approach is that one needs also the complexification of the algebra in consideration. Thus for the q - conformal algebra we can use the $U_q(sl(4))$ apparatus of Sections 4 and 5. Thus, we can propose new q - *Minkowski coordinates* as part of the appropriate q - deformed flag manifold. Using the corresponding representations and intertwiners of $U_q(sl(4))$ we can finally write down the infinite hierarchy of q - Maxwell equations.

6.2. It is well known that Maxwell equations

$$\partial^\mu F_{\mu\nu} \; = \; J_\nu \; , \quad \partial^{\mu *} F_{\mu\nu} \; = \; 0 \tag{6.1}$$

or, equivalently

$$\begin{aligned} \partial_k E_k \; = \; J_0 \; (=4\pi\rho), \quad \partial_0 E_k - \varepsilon_{k\ell m} \partial_\ell H_m \; &= \; J_k \; (=-4\pi j_k), \\ \partial_k H_k \; = \; 0 \; , \quad \partial_0 H_k + \varepsilon_{k\ell m} \partial_\ell E_m \; &= \; 0 \; , \end{aligned} \tag{6.2}$$

where $E_k \equiv F_{k0}$, $H_k \equiv (1/2)\varepsilon_{k\ell m} F_{\ell m}$, can be rewritten in the following manner:

$$\partial_k F^\pm_k \; = \; J_0 \; , \quad \partial_0 F^\pm_k \pm i\varepsilon_{k\ell m} \partial_\ell F^\pm_m \; = \; J_k \; , \tag{6.3}$$

where

$$F_k^{\pm} \equiv E_k \pm iH_k \ . \tag{6.4}$$

Not so well known is the fact that the eight equations in (6.3) can be rewritten as two conjugate scalar equations in the following way:

$$I^+ \ F^+(z) = J(z,\bar{z}) \ , \tag{6.5a}$$

$$I^- \ F^-(\bar{z}) = J(z,\bar{z}) \ , \tag{6.5b}$$

where

$$I^+ = \bar{z}\partial_+ + \partial_v - \frac{1}{2}\Big(\bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\Big)\partial_z \ , \tag{6.6a}$$

$$I^- = z\partial_+ + \partial_{\bar{v}} - \frac{1}{2}\Big(\bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\Big)\partial_{\bar{z}} \ , \tag{6.6b}$$

$$x_\pm \equiv x_0 \pm x_3, \quad v \equiv x_1 - ix_2, \quad \bar{v} \equiv x_1 + ix_2, \tag{6.7a}$$

$$\partial_\pm \equiv \partial/\partial x_\pm, \quad \partial_v \equiv \partial/\partial v, \quad \partial_{\bar{v}} \equiv \partial/\partial \bar{v}, \tag{6.7b}$$

$$F^+(z) \equiv z^2(F_1^+ + iF_2^+) - 2zF_3^+ - (F_1^+ - iF_2^+) \ , \tag{6.8a}$$

$$F^-(\bar{z}) \equiv \bar{z}^2(F_1^- - iF_2^-) - 2\bar{z}F_3^- - (F_1^- + iF_2^-) \ , \tag{6.8b}$$

$$J(z,\bar{z}) \equiv \bar{z}z(J_0 + J_3) + \bar{z}(J_1 - iJ_2) + z(J_1 + iJ_2) + (J_0 - J_3) \ , \tag{6.8c}$$

where we continue to suppress the x_μ, resp., $x_\pm, v, \bar{v}$, dependence in F and J. (The conjugation mentioned above is standard and in our terms it is : $I^+ \longleftrightarrow I^-$, $F^+(z) \longleftrightarrow F^-(\bar{z})$.)

It is easy to recover (6.3) from (6.5) - just note that both sides of each equation are first order polynomials in each of the two variables z and $\bar{z}$, then comparing the independent terms in (6.5) one gets at once (6.3).

Writing the Maxwell equations in the simple form (6.5) has also important conceptual meaning. The point is that each of the two scalar operators I^+, I^- is indeed a single object, namely it is an intertwiner of the conformal group, while the individual components in (6.1) - (6.3) do not have this interpretation. This is also the simplest way to see that the Maxwell equations are conformally invariant, since this is equivalent to the intertwining property.

Let us be more explicit. The physically relevant representations T^χ of the 4-dimensional conformal algebra $su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where n_1, n_2 are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being $(n_1+1)(n_2+1)$), and d is the conformal dimension (or energy). To these representations correspond (via Weyl's unitary trick) ERs of $Spin(5,1)$ which would be labelled as $\{(n_1 - n_2)/2\,,\, 1 + (n_1+n_2)/2\,,\, d-2\}$, cf. (1.26). Then the intertwining properties of the operators in (6.6) are given by:

$$I^+ \ :\ C^+ \longrightarrow C^0\ ,\quad I^+ \circ T^+ \ =\ T^0 \circ I^+\ , \qquad (6.9a)$$

$$I^- \ :\ C^- \longrightarrow C^0\ ,\quad I^- \circ T^- \ =\ T^0 \circ I^-\ , \qquad (6.9b)$$

where $T^a = T^{\chi^a}$, $a = 0,+,-$, $C^a = C^{\chi^a}$ are the representation spaces, and the signatures are given explicitly by:

$$\chi^+ = [2,0;2]\ ,\quad \chi^- = [0,2;2]\ ,\quad \chi^0 = [1,1;3]\ , \qquad (6.10)$$

as anticipated. Indeed, $(n_1,n_2) = (1,1)$ is the four-dimensional Lorentz representation, (carried by J_μ above), and $(n_1,n_2) = (2,0),(0,2)$ are the two conjugate three-dimensional Lorentz representations, (carried by $F^\pm_k$ above), while the conformal dimensions are the canonical dimensions of a current ($d = 3$), and of the Maxwell field ($d = 2$). We see that the variables $z, \bar z$ are related to the spin properties and we shall call them 'spin variables'. More explicitly, a Lorentz spin-tensor $G(z,\bar z)$ with signature (n_1,n_2) is a polynomial in $z,\bar z$ of order n_1, n_2, resp.

Formulae (6.9), (6.10) are part of an infinite hierarchy of couples of first order intertwiners given already in [30] for the Euclidean conformal group $SU^*(4)$, and then for the conformal group $SU(2,2)$ in [51], [15]. (Note that [30], [51] use a different approach, while [15] already uses the essential features of [16] in the context of the conformal group.) Explicitly, instead of (6.9), (6.10) we have [15] :

$$I^+_n \ :\ C^+_n \longrightarrow C^0_n\ ,\quad I^+_n \circ T^+_n \ =\ T^0_n \circ I^+_n\ , \qquad (6.11a)$$

$$I^-_n \ :\ C^-_n \longrightarrow C^0_n\ ,\quad I^-_n \circ T^-_n \ =\ T^0_n \circ I^-_n\ , \qquad (6.11b)$$

where $T^a_n = T^{\chi^a_n}$, $C^a_n = C^{\chi^a_n}$, and the signatures are:

$$\chi^+_n = [n+2,n;2]\ ,\quad \chi^-_n = [n,n+2;2]\ ,\quad \chi^0_n = [n+1,n+1;3]\ ,\quad n \in \mathbb{Z}_+\ , \qquad (6.12)$$

while instead of (6.5) we have:

$$I^+_n\ F^+_n(z,\bar z) \ =\ J_n(z,\bar z)\ , \qquad (6.13a)$$

$$I^-_n\ F^-_n(z,\bar z) \ =\ J_n(z,\bar z)\ , \qquad (6.13b)$$

where

$$I^+_n \ =\ \frac{n+2}{2}\Big(\bar z\partial_+ + \partial_v\Big) - \frac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_z\ ,\quad n \in \mathbb{Z}_+ \qquad (6.14a)$$

$$I^-_n \ =\ \frac{n+2}{2}\Big(z\partial_+ + \partial_{\bar v}\Big) - \frac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_{\bar z}\ ,\quad n \in \mathbb{Z}_+ \qquad (6.14b)$$

while $F^+_n(z,\bar z)$, $F^-_n(z,\bar z)$, $J_n(z,\bar z)$, are polynomials in $z,\bar z$ of degrees $(n+2,n)$, $(n,n+2)$, $(n+1,n+1)$, resp., as explained above. If we want to use the notation with indices as in (6.1), then $F_n^+(z,\bar z)$ and $F_n^-(z,\bar z)$ correspond to $F_{\mu\nu,\alpha_1,\dots,\alpha_n}$ which is antisymmetric in the indices μ,ν, symmetric in $\alpha_1,\dots,\alpha_n$, and traceless in every pair of indices, while $J_n(z,\bar z)$ corresponds to $J_{\mu,\alpha_1,\dots,\alpha_n}$ which is symmetric and traceless in every pair of indices. Note, however, that the analogs of (6.1) would be much more complicated if one wants to write explicitly all components. The crucial advantage of (6.13) is that the operators $I_n^\pm$ are given just by a slight generalization of $I^\pm = I_0^\pm$. In another form these operators may be obtained [51] from those for the Euclidean conformal group in [30] using Weyl's unitary trick. The Euclidean counterparts of (6.12) are χ_3^+, χ_3^-, χ_2^+, in the notation of (1.26) with $\tilde h = 3$, and $(m_1,m_2,m_3) = (0,1,n+2)$, while the Euclidean counterparts of $(6.11a,b)$ are d_2^+, d_3^+, in the notation of (1.27).

We shall call the hierarchy of equations (6.13) the **Maxwell hierarchy** . The Maxwell equations are the zero member of this hierarchy.

To proceed further we rewrite (6.14) in the following form:

$$I_n^+ = \frac{1}{2}\Big((n+2)I_1I_2 - (n+3)I_2I_1\Big) , \qquad (6.15a)$$

$$I_n^- = \frac{1}{2}\Big((n+2)I_3I_2 - (n+3)I_2I_3\Big) , \qquad (6.15b)$$

where

$$I_1 \equiv \partial_z , \quad I_2 \equiv \bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_- , \quad I_3 \equiv \partial_{\bar z} . \qquad (6.16)$$

We note in passing that group-theoretically the operators I_a correspond to the three simple roots of the root system of $sl(4)$, while the operators $I_n^\pm$ correspond to the two non-simple non-highest roots [15], [16].

This is the form that we generalize for the q - deformed case. In fact, we can write at once the general form, which follows from $(5.11a)$, $(5.12a)$ (cf. also (5.6)) :

$$_qI_n^+ = \frac{1}{2}\Big([n+2]_q I_1^q I_2^q - [n+3]_q I_2^q I_1^q\Big) , \qquad (6.17a)$$

$$_qI_n^- = \frac{1}{2}\Big([n+2]_q I_3^q I_2^q - [n+3]_q I_2^q I_3^q\Big) . \qquad (6.17b)$$

It is our task (using the previous Sections) to make this form explicit by first generalizing the variables, then the functions and the operators.

6.3. The variables $x_\pm, v, \bar v, z, \bar z$ have definite group-theoretical meaning, namely, they are six local coordinates on the flag manifold $\mathcal{Y} = SL(4)/B$, where B is the Borel subgroup of $SL(4)$ consisting of all upper diagonal matrices. (Equally well one may take the flag manifold $SL(4)/B^-$, where B^- is the Borel subgroup of lower diagonal matrices.) Under the natural conjugation (cf. also below) this is also a flag manifold of the conformal group $SU(2,2)$.

We know from Sections 3. and 4. what are the properties of the non-commutative coordinates on the $SL_q(4)$ flag manifold. We make the following identification (compare with (4.11)) :

$$x_+ = w = Y_{41}\ , \quad x_- = u = Y_{32}\ , \quad v = x = Y_{31}\ , \quad \bar{v} = y = Y_{42} \tag{6.18a}$$

$$z = \xi = Y_{21}\ , \quad \bar{z} = \eta = Y_{43}\ , \tag{6.18b}$$

for the q-Minkowski space-time coordinates and for the spin coordinates, which we denote as their classical counterparts. Thus, we obtain for the commutation rules of the q-Minkowski space-time coordinates (cf. (4.2)) :

$$\begin{aligned} x_\pm v \ &= \ q^{\pm 1} v x_\pm\ , \quad x_\pm \bar{v} \ = \ q^{\pm 1} \bar{v} x_\pm\ , \\ x_+ x_- - x_- x_+ \ &= \ \lambda v \bar{v}\ , \quad \bar{v} v \ = \ v \bar{v}\ . \end{aligned} \tag{6.19}$$

As expected, relations (6.19) coincide with the commutation relations between the translation generators P_μ of the q-conformal algebra [20]. It is also easy to notice that these relations are as the $GL_{q^{-1}}(2)$ commutation relations [45], if we identify our coordinates with the standard a, b, c, d generators of $GL_{q^{-1}}(2)$ as follows:

$$M \ = \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ = \ \begin{pmatrix} x_+ & v \\ \bar{v} & x_- \end{pmatrix}\ . \tag{6.20}$$

The q-Minkowski length is defined as the $GL_{q^{-1}}(2)$ q-determinant :

$$\ell_q \ \doteq \ \det_{q^{-1}} M \ = \ ad - qbc \ = \ x_+ x_- - q\bar{v}v\ , \tag{6.21}$$

and hence it commutes with the q-Minkowski coordinates. It has the correct classical limit $\ell_{q=1} = x_0^2 - \vec{x}^2$.

We know from (4.3) that for q phase ($|q| = 1$) the commutation relations (6.19) are preserved by an anti-linear anti-involution ω acting as :

$$\omega(x_\pm) \ = \ x_\pm\ , \quad \omega(v) \ = \ \bar{v}\ , \tag{6.22}$$

from which follows also that $\omega(\ell_q) \ = \ \ell_q$.

Remarks :

1. Note that relations (6.19) are different from the commutation relations of q-Minkowski space-time (with q real) in [11], [54], [43]. Recently, [44], it was shown that the q-Minkowski space of [11], [54], [43] can be obtained by a quantum Wick rotation (twisting) from a q-Euclidean space. The latter is also related to $GL_q(2)$, as our q-Minkowski space, however, for q real and under a different anti-linear anti-involution: $\tilde{\omega}_E(a) = d$, $\tilde{\omega}_E(b) = -q^{-1}c$, i.e., for the matrix M (cf. (6.20)) this is the unitary *, [44], while with our conjugation (6.22) M is hermitean.
2. Another proposal for deformed space-time may be obtained by extension of a new operator realization of $SU(2)$ quantum group representation matrices over non-commuting coordinates [6].
3. In the framework of algebraic field theory different proposals for quantum space-times were put forward in [42], [32].

The commutation rules of the spin variables $\bar{z}, z$ between themselves, with the q-Minkowski coordinates and with the q-Minkowski length are (cf. (4.2)) :

$$\begin{aligned}
\bar{z}z &= z\bar{z} \ , \\
x_+ z &= q^{-1} z x_+ \ , \quad x_- z = q z x_- - \lambda v \ , \\
v z &= q^{-1} z v \ , \quad \bar{v} z = q z \bar{v} - \lambda x_+ \ , \\
\bar{z} x_+ &= q x_+ \bar{z} \ , \quad \bar{z} x_- = q^{-1} x_- \bar{z} + \lambda \bar{v} \ , \\
\bar{z} v &= q^{-1} v \bar{z} + \lambda x_+ \ , \quad \bar{z}\bar{v} = q \bar{v}\bar{z} \ , \\
z \ell_q &= \ell_q z \ , \quad \bar{z} \ell_q = \ell_q \bar{z} \ .
\end{aligned} \tag{6.23}$$

Certainly, the commutation relations (6.23) are also preserved (for q phase) by the conjugation ω which acts (cf. (4.3)) by : $\omega(z) = \bar{z}$. Thus, with this conjugation $\mathcal{Y}_q$ becomes a flag manifold of $SU_q(2,2)$.

From (4.4) we know the normally ordered basis of the q - flag manifold $\mathcal{Y}_q$ considered as an associative algebra :

$$\hat{\varphi}_{ijk\ell mn} = z^i \ v^j \ x_-^k \ x_+^\ell \ \bar{v}^m \ \bar{z}^n \ , \quad i,j,k,\ell,m,n \in \mathbb{Z}_+ \ . \tag{6.24}$$

Let us denote by $\mathcal{Z}$, $\bar{\mathcal{Z}}$, and $\mathcal{M}_q$ the associative algebras with unity generated by z, $\bar{z}$, and $x_\pm, v, \bar{v}$, resp. These three algebras are subalgebras of $\mathcal{Y}_q$, and we notice the following structure of $\mathcal{Y}_q$:

$$\mathcal{Y}_q \ \cong \ \mathcal{Z} \circledS \mathcal{M}_q \circledS \bar{\mathcal{Z}} \ , \tag{6.25}$$

where $A \circledS B$ denotes the tensor product of A and B with A acting on B.

We introduce now the representation spaces C^χ , $\chi = [n_1, n_2; d]$. The elements of C^χ , which we shall call (abusing the notion) functions, are polynomials in $z, \bar{z}$ of degrees n_1, n_2, resp., and formal power series in the q - Minkowski variables. (In the general $U_q(sl(n))$ situation the signatures n_1, n_2 are complex numbers and the functions are formal power series in $z, \bar{z}$ too, cf. (3.43b).) Namely, these functions are given by:

$$\hat{\varphi}_{n_1,n_2}(\bar{Y}) = \sum_{\substack{i,j,k,\ell,m,n \in \mathbb{Z}_+ \\ i \leq n_1, \ n \leq n_2}} \mu^{n_1,n_2}_{ijk\ell mn} \ \hat{\varphi}_{ijk\ell mn} \ , \tag{6.26}$$

where $\bar{Y}$ denotes the set of the six coordinates on $\mathcal{Y}_q$. Thus the analogs of $F_n^\pm$, J_n, cf. (6.13), are :

$${}_qF_n^+ = \hat{\varphi}_{n+2,n}(\bar{Y}) \ , \quad {}_qF_n^- = \hat{\varphi}_{n,n+2}(\bar{Y}) \ , \quad {}_qJ_n = \hat{\varphi}_{n+1,n+1}(\bar{Y}) \ . \tag{6.27}$$

Next, as in (3.45), (3.46)), we introduce operators $\hat{M}_\kappa$, T_κ, where $\kappa = z$, $\pm$, v, $\bar{v}$, $\bar{z}$, and $\hat{M}_\kappa$ acts on $\hat{\varphi}_{ijk\ell mn}$ by increasing with 1 the index i,j,k,ℓ,m,n, resp., for $\kappa = z, v, -, +, \bar{v}, \bar{z}$, resp., while T_κ acts on $\hat{\varphi}_{ijk\ell mn}$ by multiplying it with q^p, where $p = i,j,k,\ell,m,n$, resp., for $\kappa = z, v, -, +, \bar{v}, \bar{z}$, resp. Then we define the

q-difference operators by (cf. (3.47)) :

$$\hat{\mathcal{D}}_\kappa \, \hat{\varphi}(\bar{Y}) \;=\; \frac{1}{\lambda} \, \hat{M}_\kappa^{-1} \, \left(T_\kappa - T_\kappa^{-1}\right) \, \hat{\varphi}(\bar{Y}) \tag{6.28}$$

Finally, we write down explicitly the operators ${}_qI_n^\pm$ in (6.17). This can be done by substituting $\hat{I}_a$ from (5.6) in our variables using (6.18), or by using directly (5.11b), (5.12b) (up to normalization and substituting our variables and representation parameters) :

$$\begin{aligned} {}_qI_n^+ \;=\; & \frac{q^2}{2} \, [n+2]_q \left(q \, \hat{\mathcal{D}}_v \, T_- \,+\, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_+ \, (T_v T_+)^{-1} \, T_{\bar{v}}\right) T_z^2 \, (T_v T_{\bar{z}})^{-1} \,- \\ & - \, \frac{1}{2} q^{-n-2} \Big(q \, \hat{M}_z \, \hat{\mathcal{D}}_v \, T_- \,+\, \hat{\mathcal{D}}_- \,+ \\ & + \, \hat{M}_z \, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_+ \, (T_v T_+)^{-1} \, T_{\bar{v}} \,+\, q^{-1} \, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_{\bar{v}} \, (T_- T_+)^{-1} \,- \\ & - \, \lambda \, \hat{M}_v \, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ \, (T_- T_+)^{-1} \, T_{\bar{v}}\Big) \, \hat{\mathcal{D}}_z \, T_z \, (T_v T_{\bar{z}})^{-1} \end{aligned} \tag{6.29a}$$

$$\begin{aligned} {}_qI_n^- \;=\; & \frac{q}{2} \, [n+2]_q \Big(q^{-1} \, \hat{\mathcal{D}}_{\bar{v}} \, (T_- T_+)^{-1} \,+\, \hat{M}_z \, \hat{\mathcal{D}}_+ \, (T_v T_+)^{-1} \, T_{\bar{v}} \,- \\ & - \, \lambda \, \hat{M}_v \, \hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ \, (T_- T_+)^{-1} \, T_{\bar{v}} \Big) \, T_- \, T_+ \, T_{\bar{v}} \, T_{\bar{z}} \,+ \\ & + \, \frac{1}{2} q^{n+3} \Big(q \, \hat{M}_z \, \hat{\mathcal{D}}_v \, T_- \,+\, \hat{\mathcal{D}}_- \,+ \\ & + \, \hat{M}_z \, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_+ \, (T_v T_+)^{-1} \, T_{\bar{v}} \,+\, q^{-1} \, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_{\bar{v}} \, (T_- T_+)^{-1} \,- \\ & - \, \lambda \, \hat{M}_v \, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ \, (T_- T_+)^{-1} \, T_{\bar{v}}\Big) \, \hat{\mathcal{D}}_{\bar{z}} \, T_- \, T_+ \, T_{\bar{v}} \end{aligned} \tag{6.29b}$$

Clearly, for $q = 1$ the operators in (6.29) coincide with (6.15).

With this the final result for the q - Maxwell hierarchy of equations is (cf. (6.27)) :

$${}_qI_n^+ \; {}_qF_n^+ \;=\; {}_qJ_n \tag{6.30a}$$

$${}_qI_n^- \; {}_qF_n^- \;=\; {}_qJ_n \tag{6.30b}$$

Note that our free q - Maxwell equations, obtained from (6.30) for $n = 0$, and ${}_qJ_0 = 0$, are different from the free q - Maxwell equations of [52], [47]. (This is natural since they use different q - Minkowski space-time from [11], [54], [43].) The advantages of our equations are: 1) they have simple indexless form; 2) we have a whole hierarchy of equations; 3) we have the full equations, and not only their free counterparts; 4) our equations are q - conformal invariant, not only q - Lorentz [47], or q - Poincaré [52], invariant. (In fact, it is not clear whether the q - Lorentz algebras of [11], [54], [43], [49] or the q - Poincaré algebra of [50] are extendable to q - conformal algebras (often easy $q = 1$ things fail for $q \neq 1$).)

6.4. The material in this subsection appeared first in [26]. We start by noting that formulae (6.13), (6.11), (6.12) are part of a much more general classification scheme (discussed in the classical Euclidean conformal group case in subsection 1.3 above, and in [15], [16]) involving also other intertwining operators, and of arbitrary order.

A subset of this scheme are two infinite two-parameter families of representations which are intertwined by the same operators (6.14). Explicitly, instead of (6.11), (6.12) we have:

$$I^+_{n_1^+,n_2^+} : C^+_{n_1^+,n_2^+} \longrightarrow C^{0+}_{n_1^+,n_2^+} , \quad I^+_{n_1^+,n_2^+} \circ T^+_{n_1^+,n_2^+} = T^{0+}_{n_1^+,n_2^+} \circ I^+_{n_1^+,n_2^+} \quad (6.31a)$$

$$I^-_{n_1^-,n_2^-} : C^-_{n_1^-,n_2^-} \longrightarrow C^{0-}_{n_1^-,n_2^-} , \quad I^-_{n_1^-,n_2^-} \circ T^-_{n_1^-,n_2^-} = T^{0-}_{n_1^-,n_2^-} \circ I^-_{n_1^-,n_2^-} \quad (6.31b)$$

where $T^a_{n_1^\pm,n_2^\pm} = T^{\chi^a_{n_1^\pm,n_2^\pm}}$, $C^a_{n_1^\pm,n_2^\pm} = C^{\chi^a_{n_1^\pm,n_2^\pm}}$, $a = \pm$, or $a = 0\pm$, and

$$\begin{aligned} \chi^+_{n_1^+,n_2^+} &= [n_1^+, n_2^+; \frac{n_1^+ - n_2^+}{2} + 1] \\ \chi^{0+}_{n_1^+,n_2^+} &= [n_1^+ - 1, n_2^+ + 1; \frac{n_1^+ - n_2^+}{2} + 2] \end{aligned} \qquad n_1^+ \in I\!N,\ n_2^+ \in Z\!\!Z_+ , \quad (6.32a)$$

$$\begin{aligned} \chi^-_{n_1^-,n_2^-} &= [n_1^-, n_2^-; \frac{n_2^- - n_1^-}{2} + 1] \\ \chi^{0-}_{n_1^-,n_2^-} &= [n_1^- + 1, n_2^- - 1; \frac{n_2^- - n_1^-}{2} + 2] \end{aligned} \qquad n_1^- \in Z\!\!Z_+,\ n_2^- \in I\!N , \quad (6.32b)$$

while instead of (6.13) in the $q = 1$ case and (6.30) in the q-deformed case, we have:

$$_qI^+_{n_1^+}\ F^+_{n_1^+,n_2^+}(z,\bar z) = J^{0+}_{n_1^+,n_2^+}(z,\bar z) , \qquad (6.33a)$$

$$_qI^-_{n_2^-}\ F^-_{n_1^-,n_2^-}(z,\bar z) = J^{0-}_{n_1^-,n_2^-}(z,\bar z) , \qquad (6.33b)$$

where $_qI^+_{n_1^+}$, $_qI^-_{n_2^-}$, are given by (5.11) (or (6.14) for $q = 1$), while $F^\pm_{n_1^\pm,n_2^\pm}(z,\bar z)$, $J^{0\pm}_{n_1^\pm,n_2^\pm}(z,\bar z)$, are polynomials in $z,\bar z$ of degrees $(n_1^\pm, n_2^\pm)$, $(n_1^\pm \mp 1, n_2^\pm \pm 1)$, resp. The Euclidean counterparts of $\chi^+_{n_1^+,n_2^+}$, $\chi^{0+}_{n_1^+,n_2^+}$, resp. are (in the notation of (1.26) with $\tilde h = 3$), χ_2^-, χ_3^-, resp., or χ_3^+, χ_2^+, resp., depending on the values of n_1^+, n_2^+, while the counterpart of (6.11a) is d_2^+, cf. (1.27). Analogously, the Euclidean counterparts of $\chi^-_{n_1^-,n_2^-}$, $\chi^{0-}_{n_1^-,n_2^-}$, resp., are χ_2^-, χ_3^+, resp., or χ_3^-, χ_2^+, resp., depending on the values of n_1^-, n_2^-, while the counterpart of (6.11b) is d_3^+.

The crucial feature which unifies these representations is the form of the operators $_qI^\pm_n$, which is not generalized anymore in equations (6.33).

We shall call the hierarchy of equations (6.33) the **generalized q - Maxwell hierarchy** . The q - Maxwell hierarchy is obtained in the partial case when $\chi^{0+}_{n_1^+,n_2^+} = \chi^{0-}_{n_1^-,n_2^-} = \chi^0_n$ which fixes three of the four parameters: $n_1^+ - 2 = n_2^+ = n_1^- = n_2^- - 2 = n$.

Another one parameter subhierarchy of the generalized q - Maxwell hierarchy is obtained if we set $n_1^\pm = n_2^\pm = n \in I\!N$, then

$$\chi^+_{n,n} = \chi^-_{n,n} = [n+1, n+1; 1] \equiv \chi^{00}_n , \qquad (6.34a)$$

$$\chi^{0\pm}_{n,n} = [n \mp 1, n \pm 1; 2] = \chi^{\mp}_{n-1} \qquad (6.34b)$$

cf. (6.12). This hierarchy will be called the **potential q - Maxwell hierarchy** . The reason is that the lowest member obtained for $n = 1$ (and $q = 1$) consists of the equations:

$$\partial_{[\mu} A_{\nu]} = F_{\mu\nu} . \qquad (6.35)$$

We also mention the equations obtained from the generalized q - Maxwell hierarchy for the minimal possible values of the parameters, namely, for $n_1^+ = n_2^- = 1$, $n_1^- = n_2^+ = 0$, i.e., the two conjugate q - Weyl equations.

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Figure Captions

Fig. 1. Partially equivalent ERs and intertwining operators for $Spin(n+1,1)$ with $n \in 2I\!N$, $\tilde{h} = n/2$.

Fig. 2. Partially equivalent ERs and intertwining operators for $Spin(n+1,1)$ with $n \in 2I\!N + 1$, $\tilde{h} = (n-1)/2$.

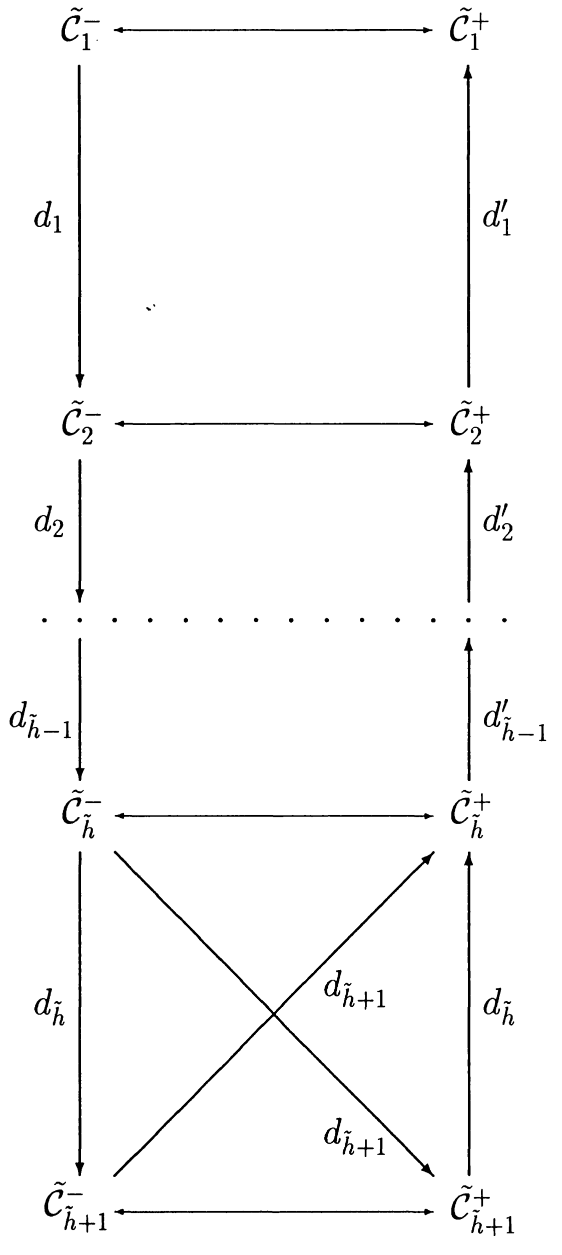
$\tilde{C}_1^-$
$\tilde{C}_1^+$
d_1
d'_1
$\tilde{C}_2^-$
$\tilde{C}_2^+$
d_2
d'_2
$d_{\tilde{h}-1}$
$d'_{\tilde{h}-1}$
$\tilde{C}_{\tilde{h}}^-$
$\tilde{C}_{\tilde{h}}^+$
$d_{\tilde{h}}$
$d_{\tilde{h}+1}$
$d_{\tilde{h}+1}$
$d_{\tilde{h}}$
$\tilde{C}_{\tilde{h}+1}^-$
$\tilde{C}_{\tilde{h}+1}^+$

Fig. 1

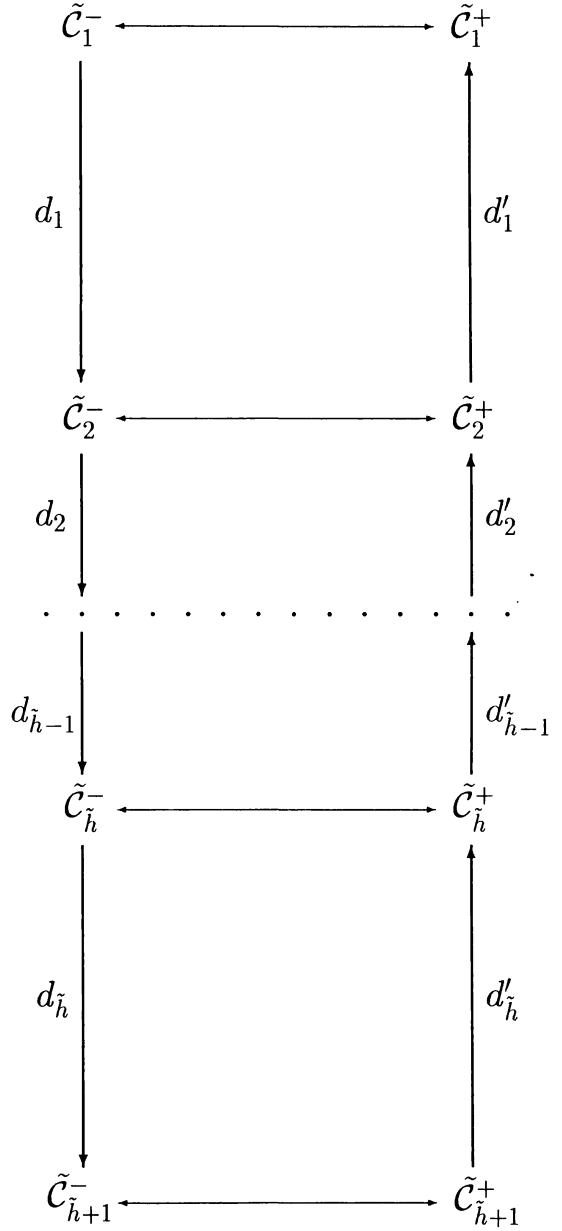
$\tilde{C}_1^-$
$\tilde{C}_1^+$
d_1
d'_1
$\tilde{C}_2^-$
$\tilde{C}_2^+$
d_2
d'_2
$d_{\tilde{h}-1}$
$d'_{\tilde{h}-1}$
$\tilde{C}_{\tilde{h}}^-$
$\tilde{C}_{\tilde{h}}^+$
$d_{\tilde{h}}$
$d'_{\tilde{h}}$
$\tilde{C}_{\tilde{h}+1}^-$
$\tilde{C}_{\tilde{h}+1}^+$

Fig. 2

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