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ON THE STRUCTURE CONSTANTS OF CERTAIN HECKE ALGEBRAS

by
Anna Helversen-Pasotto

CONTENTS

INTRODUCTION

Let $\mathbb{C}$ be the field of complex numbers, G a finite group and H a subgroup of G; let e be an idempotent in the group algebra $\mathbb{C}H$ and let $\psi: H \to \mathbb{C}$ be the character afforded by the representation of H in the left ideal $\mathbb{C}He$. Then $\mathbb{C}Ge$ affords the induced character ψ^G. By definition the Hecke algebra $\mathcal{L}(G,H,\psi)$ is the subalgebra $e\mathbb{C}Ge$ of the group algebra $\mathbb{C}G$; it is the opposite algebra of the commuting algebra $\mathrm{End}_G(\mathrm{Ind}_H^G(\lambda))$ of the induced representation $\mathrm{Ind}_H^G(\lambda)$ where λ denotes the representation of H in the left ideal $\mathbb{C}He$.

The Hecke algebra $\mathcal{L}(G,H,1_H)$ where 1_H denotes the trivial character of H is isomorphic to the algebra of functions $f: G \to \mathbb{C}$ which are constant on the (H,H) double cosets, with multiplication defined by convolution. This is the origin of the term Hecke algebra in the theory of automorphic functions.

Since Jones' discovery of his now famous polynomial (see reference /4/) there is a renewed interest in Hecke algebras and their representations.

In §1 we recall the definition of a standard basis of a Hecke algebra and of the structure constants. In §2 we give some examples of explicit computations of standard bases and structure constants of certain Hecke algebras. In §3 we study the case of the commuting algebra of the Gelfand-Graev representation of the group of two by two invertible matrices with entries in a finite field. This algebra is commutative. We give a standard basis and compute the corresponding structure constants. There arises naturally the question: Determine an explicit isomorphism of this algebra with $\mathbb{C}^d$, where d is the number of elements of the standard basis! Or determine the algebra homomorphisms of this algebra into $\mathbb{C}$ by computing their values on the standard basis! This can be done using the well-known character-table of the group GL_2 of a finite field. Via the structure constants the product of the values of two standard basis elements is expressed as a linear combination of such values. This gives rise to identities. In §4 a certain "dual" basis is introduced; this leads to two identities involving Gaussian sums over finite fields. One of them is a formal analogue of the classical Barnes' lemma in the theory of hypergeometric functions. By extrapolation via Galois theory three more identities involving Gaussian sums over certain extensions of the finite field are obtained.

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Some detailed acknowledgements concerning the results of §3 and §4 can be found in the references /2/ and /3/.

§1.- HECKE ALGEBRAS AND THEIR STRUCTURE CONSTANTS

Let us recall some of the contents of §11D and §67 of the book /1/.

PROPOSITION 1.- Let G be a finite group, H a subgroup of G and e an idempotent in the group algebra $\mathbb{C}H$, let $\psi : H \to \mathbb{C}$ be the character of the representation of H in the left ideal $\mathbb{C}He$, then $\mathbb{C}Ge$ affords the induced character ψ^G. If ζ is an irreducible character of G, then we have

$$(\zeta,\psi^G) = \zeta(e) = \dim_{\mathbb{C}}(eM) \ ,$$

where M is a $\mathbb{C}$G-module affording the character ζ and where (,) denotes the skalarproduct of characters.

DEFINITION 1.- Under the hypothesis of the preceding proposition, the subalgebra $e\mathbb{C}Ge$ of the group algebra $\mathbb{C}G$ is called the Hecke algebra $\mathfrak{X}(G,H,\psi)$ associated to G, H and ψ.

PROPOSITION 2.- Let G be a finite group, H a subgroup and $\psi: H \to \mathbb{C}^*$ a homomorphism of H into the multiplicative group $\mathbb{C}^*$ of $\mathbb{C}$. Let $x_1,\ldots,x_r$ be a system of representatives of the double cosets HxH, $x \in G$. We have $G = Hx_1H \cup \ldots \cup Hx_rH$ and $Hx_iH \cap Hx_jH = \emptyset$ for $i \neq j$, $i,j = 1,\ldots,r$. Define $e = |H|^{-1} \sum_{h \in H} \psi(h^{-1})h$, then e is an idempotent in $\mathbb{C}H$ and ψ is the character of the representation of H in the left ideal $\mathbb{C}He$. The Hecke algebra $\mathfrak{X} = \mathfrak{X}(G,H,\psi)$ can be described in the following way

(i) Define $J = \{ j \in \{1,\ldots,r\} / \psi(x_j^{-1}hx_j) = \psi(h) \text{ for every } h \in H \cap x_jHx_j^{-1} \}$;

for $j \in J$, let $\mathrm{ind}(x_j)$ denote the index of the subgroup $H \cap x_jHx_j^{-1}$ in the group H and define $a_j = \mathrm{ind}(x_j)ex_je$; the elements a_j $(j \in J)$ form a basis of $\mathfrak{X}$. If $\psi = 1$, the element a_j does not depend on the choice of the representative x_j in the double coset Hx_jH, for $j \in J$. If $\psi \neq 1$, the element a_j depends of this choice up to a factor, which is a root of unity.

(ii) For $i,j \in J$ we have $a_i a_j = \sum_{k \in J} \mu_{ijk} a_k$ where

$$\mu_{ijk} = |H| \sum_{y \in Hx_iH \cap x_kHx_j^{-1}H} a_i(y)\, a_j(y^{-1}x_k) \qquad \text{and}$$

$$a_l = \sum_{y \in G} a_l(y)\, y \quad , \quad \text{for } i,j,k,l \in J.$$

DEFINITION 2.- Under the hypothesis of the preceding proposition, the basis $(a_j)_{j \in J}$ is called the standard basis of $\mathfrak{X}$, if $\psi = 1$, and a standard basis of $\mathfrak{X}$ if $\psi \neq 1$. The complex numbers μ_{ijk} are called the structure constants of $\mathfrak{X}$, if $\psi = 1$, and the structure constants associated to the standard basis $(a_j)_{j \in J}$ if $\psi \neq 1$.

§2.- EXAMPLES

2.1.- Let F be the finite field containing q elements (q a prime-power). Set

$$S=\left\{\begin{pmatrix} c & b \\ 0 & 1\end{pmatrix} \;/\; c,b\in F,\ c\neq 0\right\},\quad U=\left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \;/\; b\in F\right\};$$

let $\psi : U\to \mathbb{C}^*$ be a homomorphism.

If ψ is not trivial, a standard basis of the Hecke algebra $\mathcal{H}(S,U,\psi)$ is given by just one element; choosing the representative $\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ in the double coset $U\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}U$ we set $a_1=\mathrm{ind}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\, e\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}e$ where e is the idempotent of $\mathbb{C}U$ corresponding to ψ, explicitly given by

$$e=|U|^{-1}\sum_{h\in U}\psi(h^{-1})\,h = q^{-1}\sum_{b\in F}\psi\begin{pmatrix} 1 & -b \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix};$$

we have $\mathrm{ind}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}=1$ and $a_1=e$. The Hecke algebra $\mathcal{H}(S,U,\psi)$ is isomorphic to $\mathbb{C}$ and a_1 is the unit element. This had to be expected, since $\mathrm{Ind}_U^S(\psi)$ is the representation of S.I. GELFAND which is known to be irreducible.

If ψ is trivial, the situation is different: the standard basis is given by $(a_c)_{c\in F-\{0\}}$, where $a_c=\mathrm{ind}\begin{pmatrix} c & 0 \\ 0 & 1\end{pmatrix}\, e\begin{pmatrix} c & 0 \\ 0 & 1\end{pmatrix}e$ and

$$e=|U|^{-1}\sum_{h\in U}\psi(h^{-1})h = q^{-1}\sum_{b\in F}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}.$$

The structure constants $\mu_{c,c',c''}$ for $c,c',c''\in F-\{0\}$ are

$$\mu_{c,c',c''}=\begin{cases} 0 & \text{if } cc'\neq c'' \\ 1 & \text{if } cc'=c''\end{cases}$$

. The Hecke algebra $\mathcal{H}(S,U,1)$ is isomorphic to $\mathbb{C}^{q-1}$.

2.2.- Let us now consider the subgroup $D=\left\{\begin{pmatrix} c & 0 \\ 0 & 1\end{pmatrix} \;/\; c\in F,\ c\neq 0\right\}$ of the group S and a homomorphism $\gamma : D\longrightarrow \mathbb{C}^*$. We have

$$D\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}D=\begin{cases} D \text{ if } b=0 \\ S-D \text{ if } b\neq 0\end{cases}$$

for $b\in F$. The elements $\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$ form a system of representatives for $D\backslash S/D$. The elements a_0 and a_1 form a standard basis of $\mathcal{H}(S,D,\gamma)$ where $a_0=\mathrm{ind}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\, e\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}e = e$ and

$$a_1=\mathrm{ind}\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\, e\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}e = (q-1)e\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}e,$$

$$e=|D|^{-1}\sum_{d\in D}\gamma(d^{-1})d = (q-1)^{-1}\sum_{c\in F,c\neq 0}\gamma\begin{pmatrix} c^{-1} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 1\end{pmatrix}.$$

The structure constants $\mu_{i,j,k}$ for $i,j,k\in\{0,1\}$ are easily deduced from

the fact that a_0 is the unit element and that $a_1^2=(q-1)a_0+(q-2)a_1$.
The Hecke algebra $\mathcal{H}(S,D,\gamma)$ is isomorphic to $\mathbb{C}^2$.

2.3.- Let G be the general linear group of invertible n by n matrices with entries in F, let B be the subgroup of upper triangular matrices and let ψ: B $\to \mathbb{C}^*$ be the trivial character. The Hecke algebra $\mathcal{H}(G,B,\psi)$ has the standard basis $(a_w)_{w\in W}$ where W is the Weylgroup; for the structure constants see /1/, § 67. This is Jones' Hecke algebra H(q,n), see /4/.

§3.- THE COMMUTING ALGEBRA OF THE GELFAND-GRAEV REPRESENTATION OF THE FINITE GROUP GL(2,F)

Let G be the group GL(2,F) of two by two invertible matrices with entries in the finite field F of q elements. Set

$$U=\left\{ \begin{pmatrix}1&b\\0&1\end{pmatrix} / b\in F\right\},\ C=\left\{\begin{pmatrix}a&0\\0&a\end{pmatrix} / a\in F,\ a\neq 0\right\},\quad H=CU\ ,$$

let α: C $\to \mathbb{C}^*$ be a homomorphism and let ψ: U $\to \mathbb{C}^*$ be a non-trivial homomorphism. Let $\alpha\psi$: H $\to \mathbb{C}^*$ be defined by $(\alpha\psi)(cu)=\alpha(c)\,\psi(u)$ for $c\in C$ and $u\in U$.

We are interested in the Hecke algebra $\mathcal{H}=\mathcal{H}(G,H,\alpha\psi)$.

We have the Bruhat decomposition

$$(3.1)\qquad G = CUD\cup CUDzU \quad\text{where}\quad D=\left\{\begin{pmatrix}c&0\\0&1\end{pmatrix} / c\in F,\ c\neq 0\right\}\ \text{and}\ z=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

The elements of $D\cup Dz$ form a system of representatives for the double cosets $H\backslash G/H$, set $x_c=\begin{pmatrix}c&0\\0&1\end{pmatrix}$ and $y_c=\begin{pmatrix}c&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&c\\1&0\end{pmatrix}$ for $c\in F$, $c\neq 0$; then $x_cHx_c^{-1}$ is equal to H; we have $(\alpha\psi)(x_c^{-1}hx_c)=(\alpha\psi)(h)$ for every $h\in H\cap x_cHx_c^{-1}$ if and only if c=1. The idempotent e of $\mathbb{C}H$ corresponding to $\alpha\psi$ is given by

$$e = q^{-1}(q-1)^{-1}\sum_{a,b\in F,a\neq 0}\alpha\begin{pmatrix}a^{-1}&0\\0&a^{-1}\end{pmatrix}\psi\begin{pmatrix}1&-b\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&a\end{pmatrix}\begin{pmatrix}1&b\\0&1\end{pmatrix}$$

and $\mathrm{ind}(x_1)ex_1e = e$ is the first element of a standard basis that we obtain following the procedure of proposition 2. Let us call this element a_0, it is the unit of $\mathcal{H}$.

For $c\in F$, $c\neq 0$ we obtain - with a little computation -
$y_cHy_c^{-1}\cap H = C$ and we have $(\alpha\psi)(y_c^{-1}hy_c) = (\alpha\psi)(h)$ for $h\in C$. Set $a_c=\mathrm{ind}(y_c)\ ey_ce$, for $c\in F$, $c\neq 0$, we have $\mathrm{ind}(y_c)=q$. The q elements a_c with $c\in F$ form a standard basis of $\mathcal{H}$. With some computation, see reference /3/, proposition 1, we obtain the structure constants $\mu_{i,j,k}$ with

$i,j,k\in F$ associated to this standard basis:

$$\mu_{0,j,k}=\begin{cases}0 & \text{if } j\neq k\\ 1 & \text{if } j=k\end{cases},\quad \mu_{i,0,k}=\begin{cases}0 & \text{if } i\neq k\\ 1 & \text{if } i=k\end{cases},\quad \mu_{i,j,0}=\begin{cases}0 & \text{if } i\neq j\\ q\,\alpha(i) & \text{if } i=j\end{cases},$$

$$\mu_{i,j,k}=\sum_{c^2=-\frac{k}{ij}}\alpha(c^{-1})\,\psi(c(i+j)-c^{-1})\qquad (c\in F,\ c\neq 0).$$

§4.- SOME ANALOGUES OF BARNES' IDENTITY FOR GAUSSIAN SUMS OVER FINITE FIELDS

Using an analogue of a Mellin transformation, we introduce a kind of dual basis of $\mathfrak{X}$. Let X be the group of multiplicative characters of the finite field F. For $\gamma\in X$ we set $a_\gamma=(q-1)^{-1}\sum_{c\in F,c\neq 0}\gamma(c)\,a_c$, note that $a_c=\sum_{\gamma\in X}\gamma(c^{-1})\,a_\gamma$, for $c\in F$, $c\neq 0$. The elements a_0 and a_γ, $\gamma\in X$, form a new basis of $\mathfrak{X}$. The new structure constants are easily deduced from reference /3/, theoreme 1, formula (5)

$$a_\beta\, a_\gamma=\frac{1}{q(q-1)}\,\delta(\alpha\beta\gamma)\;a_0+\frac{(\alpha\beta\gamma)(-1)}{q(q-1)^2}\;g(\alpha\beta\gamma)\sum_{\zeta\in X}\zeta(-1)\,g(\beta\zeta^{-1})\,g(\gamma\zeta^{-1})\,a_\zeta$$

where Gaussian sums over F and Kronecker symbols appear; these are defined by

$$g(\xi)=\sum_{c\in F,c\neq 0}\xi(c)\psi(c)\quad\text{and}\quad \delta(\xi)=\begin{cases}0 & \text{if } \xi\neq 1\\ 1 & \text{if } \xi=1\end{cases}\quad \text{for } \xi\in X.$$

In other terms, the new structure constants $\mu_{\beta,\gamma,\zeta}$ for $\beta,\gamma,\zeta\in X\cup\{0\}$ are

$$\mu_{\beta,\gamma,\zeta}=\begin{cases}\dfrac{\delta(\alpha\beta\gamma)}{q(q-1)} & \text{for } \beta,\gamma\in X,\ \zeta=0\\[2ex] \dfrac{(\alpha\beta\gamma\zeta)(-1)}{q(q-1)^2}\;g(\alpha\beta\gamma)\,g(\beta\zeta^{-1})\,g(\gamma\zeta^{-1}) & \text{for } \beta,\gamma,\zeta\in X\end{cases}$$

The charactertable of the group G is well-known, see reference /3/. If χ is the character of an irreducible representation of G, we use the same notation χ also for the extension of χ to the group algebra $\mathbb{C}G$ by linearity and we compute the restriction of this extended character χ to the subalgebra $\mathfrak{X}$ of $\mathbb{C}G$.

Using proposition 1, we know that $\chi(e)=(\mathrm{Ind}_H^G(\alpha\psi),\chi)$ for every irreducible character χ of G.

By an explicit computation, see reference /3/, proposition 2, we get $\chi^1_\mu(e)=0$, $\chi^q_\mu(e)=\delta(\alpha^{-1}\mu^2)$, $\chi^q_\mu(a_\gamma)=q^{-1}(q-1)^{-1}\delta(\alpha^{-1}\mu^2)\;\gamma(-1)\;g(\gamma\mu)^2$,

$\chi_{\mu,\nu}(e)=\delta(\alpha^{-1}\mu\nu)$, $\chi_{\mu,\nu}(a_\gamma)=q^{-1}(q-1)^{-1}\delta(\alpha^{-1}\mu\nu)(\alpha\gamma)(-1)g(\gamma\mu)g(\gamma\nu)$

$\chi_\Lambda(e)=\delta(\alpha^{-1}\lambda)$, $\chi_\Lambda(a_\gamma)=-q^{-1}(q-1)^{-1}\delta(\alpha^{-1}\lambda)(\alpha\gamma)(-1)G(\gamma^*\Lambda)$,

for $\mu,\nu,\gamma\in X$, for Λ a multiplicative character of the quadratic extension F_{q^2} of the finite field $F=F_q$ such that $\Lambda\neq\Lambda^q$; the Gaussian sum $G(\gamma^*\Lambda)$ is defined by $G(\gamma^*\Lambda)=\sum_{x\in F_{q^2},x\neq 0}\Lambda(x)\,\gamma(xx^q)\,\psi(x+x^q)$,

note that xx^q is the norm of x and that $x+x^q$ is the trace of x, for x an element of the quadratic extension of F; the multiplicative character λ is defined to be the restriction of Λ to F.

PROPOSITION 3.- Let χ be a character of G such that $\chi(e)=1$, then the extension $\chi:\mathcal{H}\longrightarrow\mathbb{C}$ is an algebra homomorphism (see /1/).

The hypothesis of proposition 3 is satisfied for $\chi=\chi^q_\mu$ with $\mu^2=\alpha$, for $\chi=\chi_{\mu,\nu}$ with $\mu\nu=\alpha$, for $\chi=\chi_\Lambda$ with $\lambda=\alpha$; for these characters χ it follows that $\chi(a_\beta)\chi(a_\gamma)=\chi(a_\beta\ a_\gamma)$ for all $\beta,\gamma\in X$.

Using the explicit computation of the structure constants $\mu_{\beta\gamma\zeta}$, we obtain the following identities

$$(q-1)^{-1}\sum_{\zeta\in X}g(\beta\zeta^{-1})g(\gamma\zeta^{-1})g(\mu\zeta)g(\nu\zeta)=\frac{g(\beta\mu)g(\beta\nu)g(\gamma\mu)g(\gamma\nu)}{g(\beta\gamma\mu\nu)}+q(q-1)\delta(\beta\gamma\mu\nu)(\mu\nu)(-1),$$

for all $\beta,\gamma,\mu,\nu\in X$, and

$$-(q-1)^{-1}\sum_{\zeta\in X}g(\beta\zeta^{-1})g(\gamma\zeta^{-1})G(\zeta^*\Lambda)=\frac{G(\beta^*\Lambda)G(\gamma^*\Lambda)}{g(\beta\gamma\lambda)}+q(q-1)\delta(\beta\gamma\lambda)\lambda(-1),$$

for all $\beta,\gamma\in X$ and for all multiplicative characters Λ of F_{q^2} such that $\Lambda\neq\Lambda^q$, here λ denotes the restriction of Λ to $F_q-\{0\}$.

For a detailed proof, see reference /3/, "Remarque 4."

The first type of identities bears a strong formal analogy to the classical Barnes' lemma in the theory of hypergeometric functions:

$$\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(b-s)\Gamma(c-s)\Gamma(m+s)\Gamma(n+s)\,ds=\frac{\Gamma(b+m)\Gamma(b+n)\Gamma(c+m)\Gamma(c+n)}{\Gamma(b+c+m+n)},$$

where b,c,m,n are complex numbers and Γ denotes the gammma function, see reference /5/.

The second type of identities is an extrapolation of the first type considering characters Λ of the extension F_{q^2} of F_q instead of the character (μ,ν) of the extension $F_q \times F_q$.

In /2/ we generalize this situation, see theorem 2, obtaining an identity for certain commutative algebras of degree 4 over the finite field F_q. This leads to five special cases, corresponding to the five conjugacy classes of the dihedral group D_4. Let us formulate the result:

THEOREM (five Barnes' identities for finite fields): Let q be a prime power, let F_q (respectively F_{q^2}, respectively F_{q^4}) denote the finite field of q (respectively q^2, respectively q^4) elements, let $\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ denote multiplicative characters of F_q, let $\Lambda, \Lambda_1, \Lambda_2$ denote multiplicative characters of F_{q^2}, let $\lambda, \lambda_1, \lambda_2$ denote respectively the restrictions of Λ, Λ_1, Λ_2 to $F_q - \{0\}$, let Φ denote a multiplicative character of F_{q^4}, let φ denote its restriction to $F_q - \{0\}$, let ε_0 be an element of F_{q^2} such that $\varepsilon_0^{q-1} = -1$, let $Tr_{2/1}$ (respectively $Tr_{4/1}$) denote the trace of F_{q^2} (respectively F_{q^4}) to F_q, let $N_{2/1}$ (respectively $N_{4/1}$) denote the norm of F_{q^2} (respectively F_{q^4}) to F_q, let ψ be a non trivial additive character of F_q, define the Kronecker symbols and Gaussian sums by

$$\delta(\alpha) = \begin{cases} 0 & \text{if } \neq 1 \\ 1 & \text{if } = 1 \end{cases} \quad , \quad g(\alpha) = \sum_{a \in F, a \neq 0} \alpha(a)\, \psi(a) \quad ,$$

$$G(\Lambda) = \sum_{x \in F_{q^2}, x \neq 0} \Lambda(x)\, \psi(Tr_{2/1}(x)) \ , \quad \mathcal{G}(\Phi) = \sum_{z \in F_{q^4}, z \neq 0} \Phi(z)\, \psi(Tr_{4/1}(z)) ,$$

then we have the following identities

$$\text{(i)} \quad \frac{g(\alpha_1\alpha_2)\, g(\alpha_2\alpha_3)\, g(\alpha_3\alpha_4)\, g(\alpha_4\alpha_1)}{g(\alpha_1\alpha_2\alpha_3\alpha_4)} = \frac{1}{q-1} \sum_{\alpha} g(\alpha_1\alpha)\, g(\alpha_2\alpha^{-1})\, g(\alpha_3\alpha)\, g(\alpha_4\alpha^{-1})$$
$$-q(q-1)\, \delta(\alpha_1\alpha_2\alpha_3\alpha_4)\, (\alpha_1\alpha_3)(-1) \cdot ,$$

(ii) $\dfrac{\mathcal{G}(\Phi^{q+1})}{g(\varphi)} = -\dfrac{1}{q+1}\sum_{\Lambda}\mathcal{G}(\Phi(\Lambda\circ N_{4/2})) - q(q-1)\,\delta(\varphi)\,\Phi(\varepsilon_0)$

(iii) $\dfrac{G(\Lambda_1\Lambda_2)\;G(\Lambda_1\Lambda_2^q)}{g(\lambda_1\lambda_2)} = \dfrac{1}{q-1}\sum_{\alpha} G(\Lambda_1(\alpha\circ N_{2/1}))\;G(\Lambda_2(\alpha^{-1}\circ N_{2/1})$

$-q(q-1)\,\delta(\lambda_1\lambda_2)\,\lambda_1(-1)$,

(iv) $\dfrac{G(\Lambda(\alpha_1\circ N_{2/1}))\quad G(\Lambda(\alpha_2\circ N_{2/1}))}{g(\alpha_1\alpha_2\lambda)} = -\dfrac{1}{q-1}\sum_{\alpha} g(\alpha_1\alpha)\,g(\alpha_2\alpha)\;G(\Lambda(\alpha^{-1}\circ N_{2/1})$

$-q(q-1)\,\delta(\alpha_1\alpha_2\lambda)\,\lambda(-1)$,

(v) $\dfrac{g(\lambda_1)\;g(\lambda_2)\;G(\Lambda_1\Lambda_2)}{g(\lambda_1\lambda_2)} = \dfrac{1}{q+1}\sum_{\Lambda,\,\lambda=1} G(\Lambda_1\Lambda)\;G(\Lambda_2\Lambda)$

$-q(q-1)\,\delta(\lambda_1\lambda_2)\,\lambda_1(-1)\;(\Lambda_1\Lambda_2)(\varepsilon_0)$;

the multiplicative character $\Lambda\circ N_{4/2}$ of F_{q^4} is defined by

$(\Lambda\circ N_{4/2})(z) = \Lambda\left(N_{4/2}(z)\right)$, for $z\in F_{q^4}$, here $N_{4/2}$ denotes the norm of F_{q^4} to F_{q^2} .

For a detailed and direct proof of this theorem see reference /2/.

Many questions arise naturally: The identity (i) corresponds to the principal series characters of the group $GL(2, F_q)$, the identity (iv) corresponds to the discrete series characters of $GL(2, F_q)$; is there a representation theoretic interpretation of the identities (ii), (iii) or (v) ? No answer is known.

Analogous computations for $GL(n,F_q)$ seem to be complicated, even the structure constants of the Hecke algebra of the Gelfand-Graev representation are not known. This algebra is commutative and its dimension is well-known.

The principal series of $GL(n,F_q)$ will probably give rise to a higher dimensional Barnes' lemma; this could lead to a higher dimensional classical Barnes' lemma and there should be interesting consequences for the theory of hypergeometric functions.

The idéntity (ii) is the most twisted in the sens of Galoistheory. Is there a generalisation to F_{q^n} for certain n instead of 4 ?

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