Antonella Cabras; Ivan Kolář; Marco Modugno

General structured bundles

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1991. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 26. pp. [105]--111.

Persistent URL: http://dml.cz/dmlcz/701483

GENERAL STRUCTURED BUNDLES

Antonella Cabras, Ivan Kolář, Marco Modugno

Abstract: A general theory of fibre bundles structured by an arbitrary differential geometric category is presented. It is proved that the structured bundles of finite type coincide with the classical associated bundles.

We develop the foundations of a general theory of fibre bundles, whose fibres are structured by an arbitrary differential geometric category. Our starting point were the classical examples of algebraic categories such as vector spaces, affine spaces, Lie groups and principal spaces, which can be treated from a unified point of view. Further examples, less known as a rule, appear in the higher order differential geometry, in particular in the jet theory, [5] , [6] .

In the first section of the paper we present the basic definitions. Then we discuss the problem of extending a functor between two categories over manifolds to the corresponding bundles and we clarify that the basic requirement is smoothness of the functor in question. The rest of the paper is devoted to the structured bundles of finite type, i.e. with finite dimensional groups of automorphisms of the fibres. Using the basic ideas of the theory of smooth structures by A. Frölicher, [3] , we deduce that the structured bundles of finite type coincide with the classical associated bundles.

We remark that a more complete treatment of some topics of the present paper, together with further related subjects, can be found in a seminar text [2] .

All manifolds and maps are assumed to be infinitely differentiable.

This paper is in final form and no version of it will be submitted for publication elsewhere.

1. CATEGORIES OVER MANIFOLDS AND STRUCTURED BUNDLES

The basic differential geometric category is the category $\underline{Mf}$ of all smooth manifolds and all smooth maps. Roughly speaking, the objects of an arbitrary differential geometric category should be manifolds with an additional structure and the morphisms should be smooth maps. The following approach is somewhat abstract, but this is a direct modification of the contemporary point of view to the concept of a concrete category, which is defined as a category over the category of sets.

Definition 1. A category over manifolds is a category $\underline{S}$ endowed with a faithful functor $\mu : \underline{S} \to \underline{Mf}$.

Manifold μF is called the underlying manifold of $\underline{S}$-object F and F is said to be an $\underline{S}$-object over μF. The assumption functor μ is faithful means that every restricted map $\mu_{F,G}: \underline{S}(F,G) \to C^{\infty}(\mu F, \mu G)$, $F,G \in \mathrm{Ob}\, \underline{S}$ is injective. Taking into account such an inclusion $\underline{S}(F,G) \subset C^{\infty}(\mu F, \mu G)$, we shall use the standard abuse of language identifying every smooth map $f: \mu F \to \mu G$ from $\mu_{F,G}(\underline{S}(F,G))$ with an $\underline{S}$-morphism $f: F \to G$.

Let $p: E \to M$ be a fibred manifold and $E_x = p^{-1}(x)$, $x \in M$.

Definition 2. An $\underline{S}$-bundle is a pair (E, σ), where $\sigma : M \to \mathrm{Ob}\, \underline{S}$ is a map satisfying $\mu\sigma(x) = E_x$ and the following condition of local triviality: for every $x \in M$ there exists a neighbourhood U, an $\underline{S}$-object F and a fibred manifold trivialization $\varphi: p^{-1}(U) \to U \times \mu F$ such that every map $\varphi_x: \sigma(x) \to F$ is an $\underline{S}$-isomorphism.

We also say that the fibred manifold $p: E \to M$ is structured in category $\underline{S}$ by means of σ.

Let $q: Z \to N$ be another fibred manifold and (Z, ρ) be another $\underline{S}$-bundle.

Definition 3. A fibred manifold morphism $f: E \to Z$ over $f_0: M \to N$ is said to be a morphism of $\underline{S}$-bundles, if every restricted map $f_x: \sigma(x) \to \rho(f_0(x))$ is an $\underline{S}$-morphism, $x \in M$.

We denote by $\underline{SB}$ the category of all $\underline{S}$-bundles and their morphisms.

Remark 1. Every local trivialization $\varphi: p^{-1}(U) \to U \times \mu F$ from Definition 2 can be called an $\underline{S}$-chart on (E, σ). Clearly, the structure of an $\underline{S}$-bundle on E can be reconstructed from every system of $\underline{S}$-charts covering E. Such an approach is studied systematically in [2] .

The classical examples of structured bundles are vector bundles, affine bundles, Lie group bundles, principal bundles and similar objects. However, in the higher order differential geometry there appear several other classes of structured bundles, which are less known. The linear tower bundles and the affine tower bundles seem to be the basic ones; they are studied in [5] and [6] .

2. BUNDLE EXTENSIONS OF SMOOTH FUNCTORS

Let $\underline{S}$ and $\bar{\underline{S}}$ be two categories over manifolds and D: $\underline{S} \to \bar{\underline{S}}$ be a functor. In the standard geometric situations, D is canonically extended into a functor $\underline{SB} \to \bar{\underline{SB}}$. We present a general construction, which clarifies that the required property of D for such an extension is smoothness in the sense of Definition 4 below. By a smoothly parametrized family of $\underline{S}$-morphisms of F into G we mean a smooth map φ: $M \times \mu F \to \mu G$, where M is a manifold, such that each restricted map φ_x: $F \to G$ is an $\underline{S}$-morphism, $x \in M$.

<u>Definition 4.</u> A functor D: $\underline{S} \to \bar{\underline{S}}$ is called smooth, if for every smoothly parametrized family of $\underline{S}$-morphisms φ: $M \times \mu F \to \mu G$ the induced family $D\varphi$: $M \times \mu DF \to \mu DG$, $(x,z) \mapsto (D\varphi_x)(z)$ is also smooth.

We remark that some authors use the word "regular" instead of "smooth" in such a situation.

Let D: $\underline{S} \to \bar{\underline{S}}$ be a smooth functor and (p: $E \to M, \mathfrak{S}$) be an $\underline{S}$-bundle. First we define the set DE as the disjoint union of $D\mathfrak{S}(x)$, $x \in M$. This yields a projection q: $DE \to M$. Every local trivialization φ: $p^{-1}(U) \to U \times \mu F$ of E induces a bijection $D\varphi$: $q^{-1}(U) \to U \times \mu DF$. Since the transition maps between any two local trivializations of $(E, \mathfrak{S})$ are smooth and D is a smooth functor, the maps $D\varphi$ induce the structure of a fibred manifold on q: $DE \to M$. Finally, we define $D\mathfrak{S}$: $M \to \mathrm{Ob}\, \bar{\underline{S}}$ by $(D\mathfrak{S})(x) = D(\mathfrak{S}(x))$. Then $(DE, D\mathfrak{S})$ is an $\bar{\underline{S}}$-bundle.

For every $\underline{S}$-bundle morphism f: $(E \to M, \mathfrak{S}) \to (\bar{E} \to \bar{M}, \bar{\mathfrak{S}})$ we define Df: $(DE, D\mathfrak{S}) \to (D\bar{E}, D\bar{\mathfrak{S}})$ by $(Df)_x = D(f_x)$, $x \in M$. Since D is a smooth functor, Df: $DE \to D\bar{E}$ is a smooth map. Thus, we have proved

<u>Proposition 1.</u> The rule $E \mapsto DE$, $f \mapsto Df$ is a functor $\underline{SB} \to \bar{\underline{SB}}$, which will be called the bundle extension of smooth functor D: $\underline{S} \to \bar{\underline{S}}$.

3. FINITE DIMENSIONAL SUBMANIFOLDS OF $C^{\infty}(M,N)$

The concept of a smooth curve on the space $C^{\infty}(M,N)$ of all smooth maps between two manifolds has a more elementary meaning than the choice of a topology on $C^{\infty}(M,N)$, [3]. We are going to show that even the concept of a finite dimensional submanifold of $C^{\infty}(M,N)$ can be reduced to the idea of a smooth curve. We shall need the following analytical result, [1]. (As usual, a smooth curve γ on a manifold M means a smooth map $\gamma : \underline{R} \to M$.)

Boman's theorem. Let M and N be manifolds and $f: M \to N$ be a map such that $f\circ\gamma$ is a smooth curve on N for every smooth curve γ on M. Then f is a smooth map.

Consider a subset $S \subset C^{\infty}(M,N)$.

Definition 5. A map $\gamma : \underline{R} \to S$ is called a smooth curve on S, if the associated map $\tilde{\gamma}: \underline{R} \times M \to N$, $\tilde{\gamma}(t,x) = \gamma(t)(x)$ is a classical smooth map. Given a manifold P, a map $f: P \to S$ is said to be smooth, if $f\circ\gamma$ is a smooth curve on S for every smooth curve γ on P.

Definition 6. A subset $S \subset C^{\infty}(M,N)$ is called a finite dimensional submanifold, if there exists a manifold Q and a bijection $i: S \to Q$ with the property that $i\circ\gamma$ is a smooth curve on Q in the classical sense if and only if γ is a smooth curve on S in the sense of Definition 5.

The bijection $i: S \to Q$ is called the defining bijection of submanifold S.

Proposition 2. Given a finite dimensional submanifold $S \subset C^{\infty}(M,N)$ with defining bijection $i: S \to Q$ and a manifold P, a map $f: P \to S$ is smooth in the sense of Definition 5 if and only if $i\circ f: P \to Q$ is a classical smooth map.

Proof. If f is smooth, then $i\circ f\circ\gamma$ is a smooth curve on Q for every smooth curve γ on P. Using Boman's theorem, we conclude that $i\circ f$ is a smooth map. Conversely, if $i\circ f$ is smooth, then $i\circ f\circ\gamma$ is a smooth curve on Q for every smooth curve γ on P by the composition property of smooth maps. Since i is the defining bijection of S, $f\circ\gamma$ is a smooth curve on S.

In particular, Proposition 2 implies that the defining bijection $i: S \to Q$ is determined up to a diffeomorphism. Hence we may identify the finite dimensional submanifold $S \subset C^{\infty}(M,N)$ with Q by means of a defining bijection $i: S \to Q$.

Proposition 3. A map $f: P \to S$ is smooth if and only if the associated map $\tilde{f}: P \times M \to N$, $\tilde{f}(u,x) = f(u)(x)$ is smooth.

Proof. If $\tilde{f}: P \times M \to N$ is smooth and $\gamma: \underline{R} \to P$ is a smooth curve, then $\widetilde{f \circ \gamma}(t,x) = f(\gamma(t))(x) = \tilde{f}(\gamma(t),x)$ is also smooth. Hence $f \circ \gamma$ is a smooth curve on S. Conversely, let $f: P \to S$ be a smooth map and $(\gamma(t), \delta(t))$ be a smooth curve on $P \times M$. Since $\gamma(t)$ is a smooth curve on P, $\widetilde{f \circ \gamma}(t,x) = \tilde{f}(\gamma(t),x): \underline{R} \times M \to N$ is a smooth map, so that $\tilde{f}(\gamma(t), \delta(t)): \underline{R} \to N$ is a smooth curve on N. By Boman's theorem, $f: P \times M \to N$ is a smooth map.

Consider the group Diff $M \subset C^{\infty}(M,M)$ of all diffeomorphisms of M.

Definition 7. If $G \subset$ Diff M is both a subgroup and a finite dimensional submanifold, then G is called a Lie subgroup of Diff M.

Our terminology is justified by the following assertion.

Proposition 4. If $G \subset$ Diff M is both a subgroup and a finite dimensional submanifold, then the multiplication map $m: G \times G \to G$, $m(g,h) = g \circ h$, the group inversion $i: G \to G$, $i(g) = g^{-1}$ and the canonical action $a: G \times M \to M$, $a(g,x) = g(x)$ of G on M are smooth maps.

Proof. If $(\gamma(t), \delta(t))$ is a smooth curve on $G \times G$, then both associated maps $\tilde{\gamma}(t,x) = \gamma(t)(x)$ and $\tilde{\delta}(t,x) = \delta(t)(x)$ are smooth. The associated map of the composition $m(\gamma(t), \delta(t))$ is $\tilde{\gamma}(t, \tilde{\delta}(t,x))$, so that it is also smooth. Hence m transforms smooth curves on $G \times G$ into smooth curves on G. By Boman's theorem, m is a smooth map. The cases of i and a are quite similar, QED.

4. STRUCTURED BUNDLES OF FINITE TYPE

Assume in the sequel that the bases of all bundles are connected. Then the fibres of every $\underline{S}$-bundle $(p: E \to M, \sigma)$ are $\underline{S}$-isomorphic. Fix an $\underline{S}$-object F isomorphic to the fibres of E and call it the standard fibre of E. Clearly, the fact the group Aut $\underline{S}(F)$ of all $\underline{S}$-automorphisms of F is a finite dimensional submanifold of Diff(μF) does not depend on the choice of F.

Definition 8. If the automorphism group $G = \mathrm{Aut}\,\underline{S}(F)$ of the standard fibre F of an $\underline{S}$-bundle (E,σ) is a finite dimensional submanifold of $\mathrm{Diff}(\mu F)$, then E is said to be an $\underline{S}$-bundle of finite type.

Hence G is a Lie group acting smoothly on F.

We are going to define the principal fibre bundle $P(E)$ of an $\underline{S}$-bundle of finite type $(p: E \to M, \sigma)$. First we consider the set $\mathrm{Iso}\,\underline{S}(F,\sigma(x))$ of all $\underline{S}$-isomorphisms from the standard fibre F into $\sigma(x)$ and we define $P(E)$ to be the disjoint union of $\mathrm{Iso}\,\underline{S}(F,\sigma(x))$, $x \in M$. This yields a projection $\pi : P(E) \to M$. Every $\underline{S}$-isomorphism $v: \sigma(x) \to F$ determines a bijection $\mathrm{Iso}\,\underline{S}(F,\sigma(x)) \to \mathrm{Aut}\,\underline{S}(F) = G$, $u \mapsto v \circ u$. This induces the structure of a smooth manifold on $\mathrm{Iso}\,\underline{S}(F,\sigma(x))$. Moreover, every local trivialization $\varphi: p^{-1}(U) \to U \times \mu F$ induces a trivialization $\bar{\varphi}: \pi^{-1}(U) \to U \times G$. By Proposition 3, the transition functions corresponding to the change of two local trivializations are smooth. This yields the structure of a fibred manifold on $P(E)$. Finally, we have a smooth right action of G on each fibre $\mathrm{Iso}\,\underline{S}(F,\sigma(x))$ defined by the composition of $\underline{S}$-morphisms. This implies

Proposition 5. $\pi : P(E) \to M$ is a principal fibre bundle with structure group G, which will be called the principal fibre bundle of E.

Example. Let (E,σ) be a vector bundle, whose standard fibre is a vector space V. If we fix a basis in V, then the linear isomorphisms from V into $\sigma(x)$ are identified with the linear bases of vector space $\sigma(x)$. Hence the principal fibre bundle $P(E)$ of E coincides with the bundle of all linear frames in the individual fibres of E.

Let $P(E)$ be the principal fibre bundle from Proposition 5. Then the fibre bundle associated to $P(E)$ with standard fibre F is the factor space $P(E) \times F/\sim$ with respect to the equivalence relation $(u,z) \sim (ug, g^{-1}z)$, $u \in P(E)$, $z \in F$, [4]. The following result is a direct consequence of our construction.

Proposition 6. The map $P(E) \times F \to E$, $(u,z) \mapsto u(z)$ identifies E with the fibre bundle associated to $P(E)$ with standard fibre F.

Hence the structured bundles of finite type coincide with the classical associated fibre bundles.

Remark 2. However, it should be underlined that the latter proposition does not imply the theory of S-bundles of finite type is reduced to the classical theory of associated fibre bundles. In a category S over manifolds there can be non-invertible morphisms. And the non-invertible morphisms cannot be reflected in the usual operations with associated bundles.

Remark 3. Another interesting class of problems in the theory of structured bundles is the study of different kinds of their prolongations. Some general ideas concerning the prolongations of structured bundles of algebraic types can be found in [2] .

REFERENCES

[1] BOMAN J. "Differentiability of a function and of its composition with a function of one variable", Math. Scand., 20 (1967), 249-268.

[2] CABRAS A., CANARUTTO D., KOLÁŘ I., MODUGNO M. "Structured bundles", Seminari Istituto di Matematica Applicata, Florence 1989, 1-99.

[3] FRÖLICHER A. "Smooth structures", Category Theory, Lecture Notes in Math., 838, 1981, 69-81.

[4] KOBAYASHI S., NOMIZU K. "Foundations of Differential Geometry, I.", New York-London-Sydney, 1963.

[5] KOLÁŘ I., MODUGNO M. "On the algebraic structure on the bundles of higher velocities", Seminari Istituto di Matematica Applicata, Florence 1989, 1-17.

[6] KOLÁŘ I., MODUGNO M. "On the algebraic structure on the jet prolongations of fibred manifolds", to appear in Czechoslovak Math. J.

ANTONELLA CABRAS, MARCO MODUGNO
ISTITUTO DI MATEMATICA
APPLICATA "G. SANSONE"
VIA S. MARTA 3
I-50139 FIRENZE

IVAN KOLÁŘ
MATHEMATICAL INSTITUTE OF
THE ČSAV BRANCH BRNO
MENDELOVO NÁM. 1,
CS-66282 BRNO