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CAYLEY TRANSFORM, OUTER EXPONENTIAL AND SPINOR NORM

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Abstract. Cayley transform of an antisymmetric $n \times n$–matrix A is the rotation matrix $U = (I + A)(I - A)^{-1}$ in $\mathbf{SO}(n)$. In the Clifford algebra the matrices A and U correspond to the bivector $\mathbf{B}$ in $\mathbf{R}^2_n$, $A\mathbf{x} = \mathbf{x} \cdot \mathbf{B}$, and to its outer exponential defined by the finite sum

$$e^{\wedge \mathbf{B}} = 1 + \mathbf{B} + \tfrac{1}{2}\mathbf{B}\wedge\mathbf{B} + \tfrac{1}{6}\mathbf{B}\wedge\mathbf{B}\wedge\mathbf{B} + \ldots$$

The outer exponential $s = e^{\wedge \mathbf{B}}$ of $\mathbf{B}$ is the unique element in the group $\mathbf{\Gamma}_n$, with real part 1, inducing the rotation U, $U\mathbf{x} = s^{-1}\mathbf{x}s$. This representation of rotations was first invented by R. Lipschitz. In this paper the above known result is given a new proof, which does not rely on indices and is therefore independent of the coordinates. The proof employes the outer and inner products only and is based on the formula $\mathbf{x}s = (\mathbf{x} + \mathbf{x} \cdot \mathbf{B})\wedge s$. The absolute value $|s|$ of s, or the spinor norm of U, is the square root of

$$s^* s = \det\left(\frac{U+I}{2}\right)^{-1},$$

where $s \to s^*$ is the reversion of the Clifford algebra.

This paper is in final form and no version of it will be submitted for publication elsewhere

1. Properties of Clifford algebras

The Clifford algebra $\mathbf{R}_n$ shall be the associative algebra over the reals $\mathbf{R}$ generated by the elements $\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_n$ subject to the relations $\mathbf{e}_i^2 = 1$ and $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$, $i \neq j$. In order to guarantee the universal property we must also require $\mathbf{e}_1\mathbf{e}_2...\mathbf{e}_n \neq \pm 1$.

$\mathbf{R}_n$ is a linear space of dimension 2^n. It is a sum of the spaces $\mathbf{R}_n^k$ each having basis elements $\mathbf{e}_{i_1i_2...i_k} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}...\mathbf{e}_{i_k}$, $1 \leq i_1 < i_2 < ... < i_k \leq n$ where $k = 0,1,...,n$ is fixed. More precicely, the basis elements are

k	$\mathbf{e}_{i_1i_2...i_k}$	
0	1	
1	$\mathbf{e}_i$	$1 \leq i \leq n$
2	$\mathbf{e}_{ij} = \mathbf{e}_i\,\mathbf{e}_j$	$1 \leq i < j \leq n$
$\vdots$	$\vdots$	
n	$\mathbf{e}_{12...n}$	

$\mathbf{R}_n^1$ shall be identified with the euclidean space $\mathbf{R}^n$. The sum of the $\mathbf{R}_n^k$ with even k will be denoted by $\mathbf{R}_n^{(0)}$, while $\mathbf{R}_n^{(1)}$ refers to odd k. $\mathbf{R}_n^{(0)}$ is a subalgebra of $\mathbf{R}_n$.

For more information about the Clifford algebras see Refs. [1], [6], [9], [10], [11].

Involutions. The Clifford algebra $\mathbf{R}_n$ has three important involutions, similar to complex conjugation. The first, called *main involution*, is the isomorphism $a \to a'$ obtained by replacing each $\mathbf{e}_i$ by $-\mathbf{e}_i$, thereby replacing each $\mathbf{a}$ in $\mathbf{R}_n^k$ by $\mathbf{a}' = (-1)^k\mathbf{a}$. By definition $(ab)' = a'b'$.

The second involution, called *reversion*, is an anti-isomorphism $a \to a^*$ obtained by reversing the order of factors $\mathbf{e}_{i_h}$ in each $\mathbf{e}_{i_1i_2...i_k}$, thereby replacing each $\mathbf{a}$ in $\mathbf{R}_n^k$ by $\mathbf{a}^* = (-1)^{[k/2]}\mathbf{a}$. By definition $(ab)^* = b^*a^*$. The third involution, called *conjugation*, is a combination of the two others $\bar{a} = a^{*\prime} = a'^*$.

Absolute value. The euclidean square norm on $\mathbf{R}^n$ extends to the whole Clifford algebra $\mathbf{R}_n$ by defining

$$|a|^2 = \Sigma\, a_{i_1i_2...i_k}^2 \qquad \text{for} \quad a = \Sigma\, a_{i_1i_2...i_k}\mathbf{e}_{i_1i_2...i_k} \qquad (a_{i_1i_2...i_k} \text{ real})$$

where the sum ranges over all ordered multi-indices $i_1i_2...i_k$ such that $1 \leq i_1 < i_2 < ... < i_k \leq n$. This gives the *absolute value* $|a|$ of a, also obtained as the square root of the real scalar part of a^*a; as an equation $|a|^2 = \mathrm{Re}(a^*a)$.

2. Unit products and spin group

Products of vectors in $\mathbf{R}^n$ are called *products* in short. The invertible products in $\mathbf{R}_n$ form the *Lipschitz group* Γ_n. If a is in Γ_n then a^*a is real and so $|a|^2 = a^*a$, from which it follows that $|ab| = |a|\,|b|$.

If $\mathbf{x}$ is in $\mathbf{R}^n$ and a is in Γ_n, then $a'^{-1}\mathbf{x}a$ is again a vector in $\mathbf{R}^n$. Furthermore, the transformation $\mathbf{x} \to a'^{-1}\mathbf{x}a$ is a euclidean isometry. In other words, for every a in Γ_n there is a matrix U_a in $\mathbf{O}(n)$ such that $a'^{-1}\mathbf{x}a = U_a(\mathbf{x})$. Conversely, every orthogonal matrix can be represented in this way. The main involution $a \to a'$ is included here in the map $\mathbf{x} \to a'^{-1}\mathbf{x}a$ in order to guarantee a coherent treatment of even-dimensional and odd-dimensional spaces.

The Lipschitz group Γ_n splits in even and odd parts $\Gamma_n = \Gamma_n^{(0)} \cup \Gamma_n^{(1)}$, where $\Gamma_n^{(i)} = \mathbf{R}_n^{(i)} \cap \Gamma_n$. The even part $\Gamma_n^{(0)}$ covers the rotation group $\mathbf{SO}(n)$ so that the unit products a, $|a| = 1$, in $\Gamma_n^{(0)}$ form a two-fold covering group $\mathbf{Spin}(n)$ of $\mathbf{SO}(n)$.

Example. For a bivector $\mathbf{B}$ in $\mathbf{R}_n^2$, $n < 6$, $(1+\mathbf{B})(1-\mathbf{B})^{-1}$ is in $\mathbf{Spin}(n)$.

Exercise. Prove that if s is in $\mathbf{Spin}(n)$ so that $1+s$ is invertible, then $\operatorname{Re}\frac{1}{1+s} = \frac{1}{2}$.

3. Outer and inner product

If two elements $\mathbf{a}$ in $\mathbf{R}_n^i$ and $\mathbf{b}$ in $\mathbf{R}_n^j$ are multiplied, then their product $\mathbf{ab}$ is in the direct sum

$$\mathbf{R}_n^{i+j} + \mathbf{R}_n^{i+j-2} + \dots + \mathbf{R}_n^{|i-j|}.$$

The component in $\mathbf{R}_n^{i+j}$ is called the *outer product* $\mathbf{a} \wedge \mathbf{b}$ and the component in $\mathbf{R}_n^{|i-j|}$ the *inner product* $\mathbf{a} \cdot \mathbf{b}$. Both products can be extended by linearity to all of $\mathbf{R}_n$. The outer product is associative $(a\wedge b)\wedge c = a\wedge(b\wedge c)$. In the graded sense the outer product is also commutative, that is,

(1) $$a\wedge b = (-1)^{ij} b\wedge a \quad \text{for } a \text{ in } \mathbf{R}_n^{(i)} \text{ and } b \text{ in } \mathbf{R}_n^{(j)}.$$

Example. If $\mathbf{x}$ is a vector and $\mathbf{B}$ a bivector, then $\mathbf{xB} = \mathbf{x}\cdot\mathbf{B} + \mathbf{x}\wedge\mathbf{B} = \frac{1}{2}(\mathbf{xB} - \mathbf{Bx}) + \frac{1}{2}(\mathbf{xB} + \mathbf{Bx})$. Also $(\mathbf{x}\cdot\mathbf{B})\wedge\mathbf{B} = \frac{1}{2}((\mathbf{x}\cdot\mathbf{B})\mathbf{B} + \mathbf{B}(\mathbf{x}\cdot\mathbf{B})) = \frac{1}{4}(\mathbf{xB}^2 - \mathbf{BxB} + \mathbf{BxB} - \mathbf{B}^2\mathbf{x}) = \frac{1}{4}(\mathbf{xB}^2 - \mathbf{B}^2\mathbf{x})$. On the other hand $\mathbf{x}\cdot(\mathbf{B}\wedge\mathbf{B}) = \frac{1}{2}(\mathbf{x}(\mathbf{B}\wedge\mathbf{B}) - (\mathbf{B}\wedge\mathbf{B})\mathbf{x}) = \frac{1}{2}(\mathbf{xB}^2 - \mathbf{B}^2\mathbf{x})$. Therefore $\frac{1}{2}\mathbf{x}\cdot(\mathbf{B}\wedge\mathbf{B}) = (\mathbf{x}\cdot\mathbf{B})\wedge\mathbf{B}$.

Exercise. If $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ and $\mathbf{x}$ are vectors, then

$$\mathbf{x} \cdot (\mathbf{a}_1{\wedge}\mathbf{a}_2{\wedge}\mathbf{a}_3{\wedge}\mathbf{a}_4) = (\mathbf{x} \cdot \mathbf{a}_1)(\mathbf{a}_2{\wedge}\mathbf{a}_3{\wedge}\mathbf{a}_4) - (\mathbf{x} \cdot \mathbf{a}_2)(\mathbf{a}_1{\wedge}\mathbf{a}_3{\wedge}\mathbf{a}_4) + (\mathbf{x} \cdot \mathbf{a}_3)(\mathbf{a}_1{\wedge}\mathbf{a}_2{\wedge}\mathbf{a}_4) - (\mathbf{x} \cdot \mathbf{a}_4)(\mathbf{a}_1{\wedge}\mathbf{a}_2{\wedge}\mathbf{a}_3).$$

Denote $\mathbf{B}_1 = \mathbf{a}_1{\wedge}\mathbf{a}_2$, $\mathbf{B}_2 = \mathbf{a}_3{\wedge}\mathbf{a}_4$ and prove $\mathbf{x} \cdot (\mathbf{B}_1{\wedge}\mathbf{B}_2) = (\mathbf{x} \cdot \mathbf{B}_1){\wedge}\mathbf{B}_2 + (\mathbf{x} \cdot \mathbf{B}_2){\wedge}\mathbf{B}_1$.

4. Outer exponential

The outer exponential of a bivector $\mathbf{B}$ in $\mathbf{R}^2_n$ is the exponential series with outer product as multiplication [7]

(2) $$e^{\wedge\mathbf{B}} = 1 + \mathbf{B} + \tfrac{1}{2}\mathbf{B}{\wedge}\mathbf{B} + \tfrac{1}{6}\mathbf{B}{\wedge}\mathbf{B}{\wedge}\mathbf{B} + \dots$$

This series is finite. The bivector $\mathbf{B}$ can be written as a sum $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \dots + \mathbf{B}_\ell$ of at most $\ell = [n/2]$ mutually orthogonal plain bivectors $\mathbf{B}_i$, $\mathbf{B}_i{\wedge}\mathbf{B}_i = 0$, where the completely orthogonal planes $\mathbf{B}_i$ have only one point in common. This decomposition is unique unless $\mathbf{B}_i^2 = \mathbf{B}_j^2$. Notwithstanding, the product

$$(1+\mathbf{B}_1){\wedge}(1+\mathbf{B}_2){\wedge}\dots{\wedge}(1+\mathbf{B}_\ell) = (1+\mathbf{B}_1)(1+\mathbf{B}_2)\dots(1+\mathbf{B}_\ell)$$

depends only on $\mathbf{B}$ and equals the outer exponential $e^{\wedge\mathbf{B}}$ of $\mathbf{B}$ [4], [5].

The reversion of $s = e^{\wedge\mathbf{B}}$ is $s^* = e^{\wedge(-\mathbf{B})}$. Since $s{\wedge}s^* = 1$, one can say that the outer inverse $s^{\wedge(-1)}$ of s equals s^*. The ordinary inverse of s is given by $s^{-1} = s^*/|s|^2$.

5. Cayley transform

An antisymmetric $n{\times}n$–matrix A is sent by the Cayley transform to the rotation matrix $U = (I+A)(I-A)^{-1}$ in $\mathbf{SO}(n)$. There corresponds a bivector $\mathbf{B}$ in $\mathbf{R}^2_n$ to A so that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{B}$ for all vectors $\mathbf{x}$ in $\mathbf{R}^n$. If $\mathbf{y} = U\mathbf{x}$, then $\mathbf{y} - A\mathbf{y} = \mathbf{x} + A\mathbf{x}$, or equivalently

(3) $$\mathbf{y} + \mathbf{B} \cdot \mathbf{y} = \mathbf{x} + \mathbf{x} \cdot \mathbf{B}.$$

Next, compute $s{\wedge}(\mathbf{y} + \mathbf{B}{\cdot}\mathbf{y}) = s{\wedge}\mathbf{y} + s{\wedge}(\mathbf{B}{\cdot}\mathbf{y})$ when $s = e^{\wedge\mathbf{B}}$. Sum up

$$\frac{1}{k!}\underbrace{(\mathbf{B}{\wedge}\mathbf{B}{\wedge}\dots{\wedge}\mathbf{B})}_{k} \wedge (\mathbf{B} \cdot \mathbf{y}) = \frac{1}{(k+1)!}\underbrace{(\mathbf{B}{\wedge}\mathbf{B}{\wedge}\dots{\wedge}\mathbf{B})}_{k+1} \cdot \mathbf{y}$$

for $k = 0, 1, 2, \ldots$ to obtain $s\wedge(\mathbf{B}\cdot\mathbf{y}) = (s-1)\cdot\mathbf{y}$. Since $s\wedge\mathbf{y} + (s-1)\wedge\mathbf{y} = s\mathbf{y}$, it follows that $s\wedge(\mathbf{y} + \mathbf{B}\cdot\mathbf{y}) = s\mathbf{y}$. Similarly, $s\wedge(\mathbf{x} + \mathbf{x}\cdot\mathbf{B}) = s\wedge\mathbf{x} - (s-1)\wedge\mathbf{x} = \mathbf{x}\wedge s + \mathbf{x}\cdot(s-1) = \mathbf{x}s$. Therefore, the equation (3) is equivalent to

(4) $$s\,\mathbf{y} = \mathbf{x}\,s$$

or $U\mathbf{x} = s^{-1}\mathbf{x}s$. This representation of rotations was first invented by R. Lipschitz. To pay hommage to him we have denoted the Lipschitz group by $\mathbf{\Gamma}$, a mirror image of $\mathbf{L}$.

All told we have sketched a novel proof for a previously known result [5], [9], [13].

Theorem. An antisymmetric $n{\times}n$–matrix A and the rotation matrix $U = (I+A)(I-A)^{-1} \in \mathbf{SO}(n)$ correspond, respectively, to the bivector $\mathbf{B} \in \mathbf{R}^2_n$, $A\mathbf{x} = \mathbf{x}\cdot\mathbf{B}$, and to its outer exponential $s = e^{\wedge\mathbf{B}} \in \mathbf{\Gamma}^{(0)}_n$, which is the unique element of $\mathbf{\Gamma}^{(0)}_n$, with real part 1, inducing the rotation U, $U\mathbf{x} = s^{-1}\mathbf{x}s$. Conversely, every rotation U in $\mathbf{SO}(n)$, with eigenvalues different from -1, is <u>uniquely</u> obtained in this way.

The following table gives two different kinds of connections between the rotation and spin groups

$\mathbf{SO}(n)$	$\mathbf{Spin}(n)$
$e^{\mathbf{A}}$	$\pm\, e^{\mathbf{B}/2}$
$\frac{\mathbf{I}+\mathbf{A}}{\mathbf{I}-\mathbf{A}}$	$\pm\frac{e^{\wedge\mathbf{B}}}{\lvert e^{\wedge\mathbf{B}}\rvert}$

Absolute value of outer exponential. If a rotation matrix U in $\mathbf{SO}(n)$ does not rotate any plane by a half-turn, then there is a unique element s in $\mathbf{\Gamma}^{(0)}_n$, with real part 1, so that $U\mathbf{x} = s^{-1}\mathbf{x}s$. The absolute value $|s|$ of s is the square root of s^*s, which equals [12], [13]

(5) $$\det(I-A) = \det\left(\frac{U+I}{2}\right)^{-1}$$

where $A = (U-I)(U+I)^{-1}$. The absolute value is also the square root of

$$s^*s = (1-\mathbf{B}_1^2)(1-\mathbf{B}_2^2)\ldots(1-\mathbf{B}_\ell^2).$$

Combined rotations. Take two antisymmetric matrices A_1, A_2 and the corresponding rotations U_1, U_2 as well as bivectors $\mathbf{B}_1, \mathbf{B}_2$ and their outer exponentials $s_1 = e^{\wedge \mathbf{B}_1}, s_2 = e^{\wedge \mathbf{B}_2}$. Then also $s_1 \wedge s_2 = e^{\wedge(\mathbf{B}_1+\mathbf{B}_2)}$ is in $\mathbf{\Gamma}_n^{(0)}$. If the combined rotation U_2U_1 does not have -1 as its eigenvalue, or equivalently, is represented by such an element s_1s_2 in $\mathbf{\Gamma}_n^{(0)}$ that $\mathrm{Re}(s_1s_2) \neq 0$, then the matrix identity [12]

$$\frac{U_2U_1+I}{2} = \frac{U_2+I}{2}(I+A_2A_1)\frac{U_1+I}{2}$$

shows that $\lambda = 1/\mathrm{Re}(s_1s_2)$ is a solution of the quadratic equation

(6) $\qquad \lambda^2 = \det(I+A_2A_1)^{-1}.$

In other words, when multiplied by the scalar λ the product s_1s_2 is sent to the outer exponential of a unique bivector $\lambda s_1 s_2 = e^{\wedge \mathbf{B}}$. This is the reason why the spinor norm was introduced in the first place [2], [12], [13]. See also Refs. [3], [8], [9].

Remark. It is important to observe that the set of rotations U, $\det(U+I) \neq 0$, represented by the products $s = e^{\wedge \mathbf{B}}$, which are expressed in terms of the outer product only, does not depend on the scalar product of the underlying vector space. However, when writing down the actual rotation $U\mathbf{x} = s^{-1}\mathbf{x}s$, the inner product is also employed. Therefore, it might be interesting to know the effect of the quadratic form on the rotation and spin groups.

6. Indefinite quadratic forms

The Clifford algebra $\mathbf{R}_{p,q}$ shall be the associative algebra over the reals $\mathbf{R}$ generated by the elements $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ subject to the relations

$$\begin{array}{ll} \mathbf{e}_i^2 = 1 & 1 \leq i \leq p \\ \mathbf{e}_i^2 = -1 & p+1 \leq i \leq p+q = n \\ \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i & i < j. \end{array}$$

In order to guarantee the universal property we must also require $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n \neq \pm 1$.

The main differences between the positive definite case $\mathbf{R}_n = \mathbf{R}_{n,0}$ and the other Clifford algebras $\mathbf{R}_{p,q}$ will be reviewed in the following. In the Clifford algebra $\mathbf{R}_{p,q}$ the quadratic forms $a \to \mathrm{Re}(a^*a)$ and $a \to \mathrm{Re}(\bar{a}a)$ are usually non-definite. The Lipschitz group $\mathbf{\Gamma}_{p,q}$ consists of all elements in $\mathbf{R}_{p,q}$ which can be written as products of non-isotropic vectors $\mathbf{x}$, $\mathbf{x}^2 \neq 0$, in $\mathbf{R}_{p,q}^1 = \mathbf{R}^{p,q}$. For a product u in

$\mathbf{\Gamma}_{p,q}$ the expression $\bar{u}u$ is always real. A *unit* product u satisfies $\bar{u}u = \pm 1$. The unit products form a subgroup $\mathbf{Pin}(p,q)$ of $\mathbf{\Gamma}_{p,q}$. The even Lipschitz group $\mathbf{\Gamma}^{(0)}_{p,q}$ has a subgroup of even unit products $\mathbf{Spin}(p,q) = \mathbf{R}^{(0)}_{p,q} \cap \mathbf{Pin}_{p,q}$, which further has a subgroup $\mathbf{Spin}^{+}(p,q)$ where $\bar{u}u = 1$. Since $\mathbf{R}^{(0)}_{p,q} \simeq \mathbf{R}^{(0)}_{q,p}$ we also have the isomorphisms $\mathbf{Spin}(p,q) \simeq \mathbf{Spin}(q,p)$. The groups $\mathbf{Pin}(p,q)$, $\mathbf{Spin}(p,q)$ and $\mathbf{Spin}^{+}(p,q)$ are two-fold coverings of the matrix groups $\mathbf{O}(p,q)$, $\mathbf{SO}(p,q)$ and $\mathbf{SO}^{+}(p,q)$, which is the identity component of $\mathbf{SO}(p,q)$. Also the group $\mathbf{Spin}^{+}(p,q)$ is connected with the following exceptions $\mathbf{Spin}^{+}(0,0) = \mathbf{Spin}^{+}(1,0) = \mathbf{Spin}^{+}(0,1) = \pm 1$ and $\mathbf{Spin}^{+}(1,1) = \{x + y\mathbf{e}_{12} \mid x^2 - y^2 = 1\}$ [9, p. 427].

Every linear isometry L of $\mathbf{R}^{p,q}$, connected with the identity of $\mathbf{SO}(p,q)$, is the exponential of an antisymmetric transformation A of $\mathbf{R}^{p,q}$, $L = e^{A}$, if and only if $\mathbf{R}^{p,q}$ is one of the following $\mathbf{R}^{n} = \mathbf{R}^{n,0}$, $\mathbf{R}^{0,n}$, $\mathbf{R}^{p,1}$ or $\mathbf{R}^{1,q}$ [10, pp. 150-152]. In the same orthogonal spaces there is a bivector $\mathbf{B} \in \mathbf{R}^{2}_{p,q}$ such that

$$L\mathbf{x} = e^{-\mathbf{B}}\mathbf{x}e^{\mathbf{B}}$$

for any vector $\mathbf{x} \in \mathbf{R}^{p,q}$ [10, p. 160].

Finally, given a bivector $\mathbf{B}$ in $\mathbf{R}^{2}_{p,q}$ one can, in general, find other bivectors $\mathbf{F}$ such that $e^{\mathbf{B}} = -e^{\mathbf{F}}$, and hence $e^{-\mathbf{B}}\mathbf{x}e^{\mathbf{B}} = e^{-\mathbf{F}}\mathbf{x}e^{\mathbf{F}}$ for all vectors $\mathbf{x}$ in $\mathbf{R}^{p,q}$. The only exceptions concern the following cases [10, p. 172]:

$\mathbf{R}^{1,1}$ for all $\mathbf{B}$
$\mathbf{R}^{1,2}$ and $\mathbf{R}^{2,1}$ for all $\mathbf{B} \neq 0$ such that $\mathbf{B}^2 \geq 0$
$\mathbf{R}^{1,3}$ and $\mathbf{R}^{3,1}$ for all $\mathbf{B} \neq 0$ such that $\mathbf{B}^2 = 0$.

The outer exponential $e^{\wedge\mathbf{B}}$ of the bivector $\mathbf{B}$ in $\mathbf{R}^{2}_{p,q}$ need not be invertible, that is, it does not necessarily belong to the Lipschitz group $\mathbf{\Gamma}_{p,q}$. However, an invertible $s = e^{\wedge\mathbf{B}}$ is in $\mathbf{\Gamma}_{p,q}$. If the mutually commuting plain bivectors in the orthogonal decomposition $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \ldots + \mathbf{B}_\ell$ satisfy $\mathbf{B}_i^2 \neq 1$, then $\bar{s}s = (1-\mathbf{B}_1^2)(1-\mathbf{B}_2^2)\ldots(1-\mathbf{B}_\ell^2) \neq 0$, and the product $s = (1+\mathbf{B}_1)(1+\mathbf{B}_2)\ldots(1+\mathbf{B}_\ell)$ is invertible.

In the orthogonal space $\mathbf{R}^{p,q}$ an antisymmetric transformation A, $\det(I - A) \neq 0$, corresponds to the rotation $U = (I + A)(I - A)^{-1} \in \mathbf{SO}(p,q)$, $\det(U + I) \neq 0$.

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