ON THE PRODUCTS OF QUANTUM LOGICS

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A definition of a product of quantum logics is formulated and a comparison with the free orthodistributive product of orthomodular σ-lattices is given.

A quantum logic is the couple (L,M) , where L is an orthomodular σ-lattice and M is a set of states (i.e. probability measures) on L , which is strong for L , i.e. the statement

$\{m \in M : m(a) = 1\} \subset \{m \in M : m(b) = 1\}$

implies that $a \leq b$, $a,b \in L$. This notion was introduced by Gudder [4] . The physical interpretation of (L,M) is as follows. The set L is interpreted as the set of all experimentally verifiable propositions of a physical system(the " logic" of the system) , and M is the set of physical states. The requirement of the existence of a strong set of states restricts the choice of orthomodular σ-lattices suited for description of physical systems, there are orthomodular σ-lattices with no states[3]. We shall also suppose the Jauch - Piron property, i.e.

$m(a_i) = 1$ for all $i \in N$ iff $m(\bigwedge_{i \in N} a_i) = 1$

for any $m \in M$. More details on quantum logicd can be found in [6] and [9].

To describe a physical system which is composed of two other systems, we need a kind of the product of quantum logics. In the traditional approach to quantum theory, it is supposed that to any physical system there is a Hilbert space (complex, separable, with the dimension at least three) . The set of propositions is the lattice $L(H)$ of all closed linear subspaces of H and states are represented by the density operators. The joint physical system consisting of two other systems is then described by the tensor product of the Hilbert spaces of these two systems.

A product of orthomodular σ-lattices was defined in the following way [5]. We recall that the elements a,b of an orthomodular lattice L are compatible $(a \leftrightarrow b)$ if

$$a = (a \wedge b) \vee (a \wedge b^{\perp}) .$$

Definition 1 . Let $\mathcal{C}$ be a subcategory of the category of orthomodular σ-lattices. Let $\{L_i : i \in I\}$ and L be elements of $\mathcal{C}$. Then $(L, (u_i)_{i \in I})$ is a tensor product (or free orthodistributive product) of the L_i's if

(i) $u_i : L_i \to L$ are injections in $\mathcal{C}$, $i \in I$,

(ii) $\bigcup_{i \in I} u_i (L_i)$ generates L ,

(iii) for any at moust countable subset F of I , $\bigwedge_{i \in F} u_i (a_i) = 0$ for $a_i \in L_i$ iff at least one a_i is zero,

(iv) $u_i (a_i) \leftrightarrow u_j (a_j)$ for any $i,j \in I$, $i \neq j$.

In the category of Hilbert space lattices, it was shown [1] ,[5] that if H_1 , H_2 are complex, separable, of the dimension at least three, then there are exactly two (unequivalent) products, defined by

$$\text{(i)} \quad \begin{aligned} u_1 &: L(H_1) \to L(H_1 \otimes H_2) \\ &\quad P \mapsto P \otimes H_2 \\ u_2 &: L(H_2) \to L(H_1 \otimes H_2) \\ &\quad P \mapsto H_1 \otimes P \end{aligned}$$

and

$$\text{(ii)} \quad \begin{aligned} u_1 &\quad L(H_1) \to L(\overline{H}_1 \otimes H_2) \\ &\quad P \mapsto \overline{P}_1 \otimes H_2 \\ u_2 &\quad L(H_2) \to L(\overline{H}_1 \otimes H_2) \\ &\quad P \mapsto \overline{H}_1 \otimes P \end{aligned}$$

where $H_1 \otimes H_2$ is the tensor product of Hilbert spaces H_1 and H_2 , and $\overline{H}$ is the dual of H .

In the case of real Hilbert spaces there is exactly one product defined by (i) .

We shall introduce a definition of a product of quantum logics. We need some preliminary remarks. Let S be a set of states on a logic L . We say that a state p on L is a superposition of the states in S if $S(a) = 1$ implies $p(a) = 1$ for $a \in L$, where $S(a) = 1$ means that $s(a) = 1$ for any $s \in S$ [9] . If (L,M) is a quantum logic, we shall write $\overline{S} = \{p \in M : S(a) = 1 \Rightarrow p(a) = 1\}$ for any $S \subset M$.

Definition 2 . Let (L_1 ,M_1) , (L_2 ,M_2) , (L,M) be quantum logics. We shall say that $(L,M)_{\alpha_1 \beta}$ is the tensor product of (L_i,M_i), i

if

(i) $\alpha : L_1 \times L_2 \longrightarrow L$, $\beta : M_1 \times M_2 \longrightarrow M$

$\beta(m_1, m_2)(\alpha(a_1, a_2)) = m_1(a_1) . m_2(a_2)$

for any $m_i \in M_i$, $a_i \in L_i$, $i = 1,2$,

(ii) $\overline{\{m \in M : m(a) = 1\}} = \overline{\{\beta(m_1, m_2) : \beta(m_1, m_2)(a) = 1\}}$

for elements $a \in L$ of the form

$a = \bigwedge_k \alpha(a_1^k, a_2^k)$, $a_1^k \in L_1$, $a_2^k \in L_2$, $k \in N$,

and

$a = \alpha(a_1, 1)^{\perp}$, resp. $a = \alpha(1, a_2)^{\perp}$, $a_1 \in L$, $a_2 \in L_2$,

(iii) $\alpha[L_1 \times L_2]$ generates L,

(iv) $\overline{\beta[M_1 \times M_2]} = M$.

Theorem 1. Let $(L,M)_{\alpha,\beta}$ be a tensor product of (L_1, M_1) and (L_2, M_2). Let us put

$u_1 : L_1 \to L$, $u_2 : L_2 \to L$

$a \mapsto \alpha(a, 1)$ $\qquad a \mapsto \alpha(1, a)$.

Then (L, u_1, u_2) is a free product of L_1, L_2 by Def 1.

Proof. By (i) of Def. 2 we have

$\beta(m_1, m_2)(\alpha(1,1)) = m_1(1).m_2(1) = 1$

for any $m_i \in M_i$, $i = 1,2$. From this we get $m(\alpha(1,1)) = 1$ for all $m \in \overline{\beta[M_1 \times M_2]}$, and by (iv), $m(\alpha(1,1)) = 1$ for all $m \in M$, i.e. $\alpha(1,1) = u_1(1) = u_2(1) = 1$.

For any $a \in L_1$, $\beta(m_1, m_2)(\alpha(a^{\perp}, 1)) = m_1(a^{\perp}).m_2(1) = 1$ $= (1 - m_1(a)) . m_2(1) = 1 - m_1(a).m_2(1) = 1 - \beta(m_1, m_2)(\alpha(a,1)) =$ $= \beta(m_1, m_2)(\alpha(a,1)^{\perp})$ for all $m_i \in M_i$, $i = 1,2$. From this we obtain

$\overline{\{\beta(m_1, m_2) : \beta(m_1, m_2)(\alpha(a^{\perp}, 1)) = 1\}} =$

$\overline{\{\beta(m_1, m_2) : \beta(m_1, m_2)(\alpha(a,1)^{\perp}) = 1\}}$,

which implies by (ii) that $\alpha(a^{\perp}, 1) = \alpha(a,1)^{\perp}$, i.e. $u_1(a^{\perp}) = u_1(a)^{\perp}$. Similarly, $u_2(a^{\perp}) = u_2(a)^{\perp}$, $a_2 \in L_2$.

By the Jauch - Piron property we have

$\beta(m_1, m_2)(\alpha(\bigwedge_k a_1^k, 1)) = 1$ iff $m_1(\bigwedge_k a_1^k) = 1$ iff $\beta(m_1, m_2)(\bigwedge_k \alpha(a_1^k, 1)) = 1$. From this we obtain

$\overline{\{\beta(m_1, m_2) : \beta(m_1, m_2)(\alpha(\bigwedge_k a_1^k, 1)) = 1\}} =$

$= \overline{\{\beta(m_1, m_2) : \beta(m_1, m_2)(\bigwedge_k(\alpha(a_1^k, 1)) = 1\}}$,

which implies by (ii) that $\alpha(\bigwedge_k a_1^k, 1) = \bigwedge_k \alpha(a_1^k, 1)$, i.e. $u_1(\bigwedge_k a_1^k) = \bigwedge_k u_1(a_1^k)$. This shows that u_1 and u_2 are orthohomomorphisms.

Now $u_1(a) = u_1(a')$, $a', a \in L_1$ implies that

$\beta(m_1, m_2)(\alpha(a, 1)) = \beta(m_1, m_2)(\alpha(a', 1))$ for any $m_i \in M_i$, $i = 1,2$, which implies that $m_1(a) = m_1(a')$ for any

$m_1 \in M_1$, i.e. $a = a'$. Hence u_1 and u_2 are injections.

For any $m_i \in M_i$, $i = 1,2$, we have by the Jauch - Piron property,

$\beta(m_1, m_2)(u_1(a_1) \wedge u_2(a_2)) = \beta(m_1, m_2)(\alpha(a_1, 1) \wedge \alpha(1, a_2)) = 1$ iff $\beta(m_1, m_2)(u_1(a_1)) = 1$ and $\beta(m_1, m_2)(u_2(a_2)) = 1$ iff $m_1(a_1) = 1$ and $m_2(a_2) = 1$ iff $\beta(m_1, m_2)(\alpha(a_1, a_2)) = 1$, i.e.

$$\{\beta(m_1, m_2) : \beta(m_1, m_2)(\alpha(a_1, 1) \wedge \alpha(1, a_2)) = 1\}^- =$$
$$= \{\beta(m_1, m_2) : \beta(m_1, m_2)(\alpha(a_1, a_2)) = 1\}^- ,$$

hence by (ii) $u_1(a_1) \wedge u_2(a_2) = \alpha(a_1, a_2)$. This shows (ii) of Def. 1. Now let $a_1 \in L_1$, $a_1 \neq 0$ and $u_1(a_1) \wedge u_2(a_2) = 0$, $a_2 \in L_2$. Let $m_1^0 \in M_1$ be such that $m_1^0(a_1) = 1$ (the existence of m_1^0 follows from the fact that M_1 is strong for L_1).Then

$$\beta(m_1^0, m_2)(u_1(a_1) \wedge u_2(a_2)) = \beta(m_1^0, m_2)(\alpha(a_1, a_2)) =$$
$$= m_1^0(a_1) m_2(a_2) = 0 \text{ iff } m_2(a_2) = 0 .$$

Thus $u_1(a_1) \wedge u_2(a_2) = 0$ implies $m_2(a_2) = 0$ for any $m_2 \in M$, i.e. $a_2 = 0$.

Finally, for any $a_i \in L_i$, $i = 1,2$, we have

$$\beta(m_1, m_2)(u_1(a_1) \wedge u_2(a_2)) = \beta(m_1, m_2)(\alpha(a_1, a_2)) =$$
$$= m_1(a_1) m_2(a_2) = \beta(m_1, m_2)(u_1(a_1))\, \beta(m_1, m_2)(u_2(a_2))$$

for any $m_i \in M_i$ $i = 1,2$. This implies that $u_1(a_1)$ and $u_2(a_2)$ are independent (in the probabilistic sense), and by [2] they have joint probability distributions in all states of $\beta[M_1 \times M_2]^- = M$, hence they are compatible. This completes the proof.

Let H be a real or complex separable Hilbert space, $\dim H \geq 3$. If we put $M = \{m_\varphi : \varphi \in H, \|\varphi\| = 1\}$, where m_φ is the vector state corresponding to the vector φ by the Gleason theorem [9], then $(L(H), M)$ is a quantum logic. Let $(L(H_1), M_1)$, $(L(H_2), M_2)$, $(L(H_1 \otimes H_2), M)$ and $L(\overline{H}_1 \otimes H_2, \overline{M})$ be quantum logics of the corresponding Hilbert spaces. If we put

$$\text{(i) } \alpha : \quad (P_1, P_2) \longrightarrow P_1 \otimes P_2 \;, \; P_i \; L(H_i) \;, \; i = 1,2 \;,$$
$$\beta : \quad (m_{\varphi_1}, m_{\varphi_2}) \longmapsto m_{\varphi_1 \otimes \varphi_2} \quad \varphi_i \in H_i \;, \; i = 1,2 \;,$$

or

$$\text{(ii) } \overline{\alpha} : \quad (P_1, P_2) \longrightarrow \overline{P}_1 \otimes P_2 \;, \; P_i \in L(H_i) \;, \; i = 1,2$$
$$\overline{\beta} : \quad (m_{\varphi_1}, m_{\varphi_2}) \longmapsto m_{\overline{\varphi}_1 \otimes \varphi_2} \quad \varphi_i \in H_i \;, i = 1,2 \;,$$

then it can be easily checked that $(L(H_1 \otimes H_2), M)_{\alpha,\beta}$ and $(L(\overline{H}_1 \otimes H_2), \overline{M})_{\overline{\alpha},\overline{\beta}}$ are the products of $(L(H_i), M_i)$, $i = 1,2$. More details on the products of quantum logics are in [7] and [8] .

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