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SOME REMARKS ON THE COHOMOLOGY OF THE POISSON ALGEBRA

by P.B.A. LECOMTE

Introduction

1. In this paper, we illustrate in the case of the Poisson algebra a method designed for computing the Chevalley cohomology of the Lie algebras associated to manifolds.

We will insist more on some tools of the method rather than on the results. For instance, we will digress a little on the symbols of multilinear differential operators which are not only basic in the computation of the cohomology but also are useful in other questions.

The cohomology of the Poisson algebra is far from being known. The more general result has been obtained by Vey [6] : the cohomology is isomorphic to the tensor product of the de Rham cohomology of some principal bundle whose structure group is a maximal compact subgroup of the symplectic group and of some cohomology related to the formal symplectic vector fields on $\mathbb{R}^{2n}$ and which is not completely known. Another important result, because of its use in the theory of $\star$-products, is the description of the second and the third cohomology spaces [3]. Here, we will indicate how to compute the cohomology of the 2 and 3-differentiable cochains.

The method and the results in this paper have been obtained together with M. De Wilde. A detailed version including proofs will appear elsewhere [2].

Symbols of differential operators

2. For simplicity, we define only the symbol of a p-linear differential operator on the space N of C_∞-functions of a manifold M. Such an operator is a p-linear mapping $C : N^p \to N$ whose local expression in any chart $(U,(x^1,\dots,x^m))$ of M reads

$$C(u_0,\dots,u_{p-1})\,|_U = \sum_{|\alpha_0|+\dots+|\alpha_{p-1}|\leq k} C_{\alpha_0,\dots,\alpha_{p-1}}\, D_x^{\alpha_0}u_0\dots D_x^{\alpha_{p-1}}u_{p-1}$$

where the $C_{\alpha_0,\dots,\alpha_{p-1}}$'s are C_∞-functions on U, and, for $\alpha = (t_1,\dots,t_m)$,

$$D_x^\alpha u \quad \text{means} \quad D_{x^1}^{t_1} \dots D_{x^m}^{t_m} u \ .$$

Under a change of coordinates, the above local expression behaves according to a rather complicated rule which has no useful meaning except for the sum of the terms whose order $\Sigma|\alpha_i|$ is maximal. Indeed, to perform in that sum substitutions of the form

$$D_x^\alpha u \to \xi^\alpha = (\xi^1)^{t_1} \dots (\xi^m)^{t_m} \quad (\xi \in T_x^\star M)$$

gives a well-defined intrinsic homogeneous polynomial in $\xi_0, \dots, \xi_{p-1} \in T_x^\star M$:

$$\sigma_C(\xi_0, \dots, \xi_{p-1}) = \sum_{|\alpha_0| + \dots + |\alpha_{p-1}| = k} C_{\alpha_0, \dots, \alpha_{p-1}} \xi_0^{\alpha_0} \dots \xi_{p-1}^{\alpha_{p-1}}$$

which is called the *total symbol of* C. It will be also useful to consider the *symbol* $\bar{\sigma}_C$ of C, which is defined as the component of σ_C of maximal lexicographical degree. Recall that the lexicographical degree of

$$C_{\alpha_0, \dots, \alpha_{p-1}} \xi_0^{\alpha_0} \dots \xi_{p-1}^{\alpha_{p-1}}$$

is $\vec{r} = (r_o = |\alpha_o|, \dots, r_{p-1} = |\alpha_{p-1}|)$ and that $\vec{r} > \vec{r}\,'$ means

$$\exists \ i < p : \vec{r}\,' = (r_o, \dots, r_{i-1}, r_i', \dots, r_{p-1}') \quad \text{and} \quad r_i' < r_i$$

3. An important feature of the notion of symbols is the composition formula which allows easy manipulations of differential operators : as easily seen the total symbol of

$$C' = C(C_o(u_1, \dots), \dots, C_{p-1}(v_1, \dots))$$

is given by

$$\sigma_{C'} = \sigma_C(\sum_i \xi_i, \dots, \sum_j \eta_j) \sigma_{C_o}(\xi_1, \dots) \dots \sigma_{C_{p-1}}(\eta_1, \dots) .$$

Let us illustrate the use of this formula by computing the derivations of the Poisson algebra (see § 5 for notations), that is the linear operators $D : N \to N$ such that

$$D\{u, v\} = \{Du, v\} + \{u, Dv\}.$$

The symbol of $\{,\}$ being $\Lambda(\xi,\eta)$, it follows for σ_D the equation

$$\sigma_D(\xi+\eta)\Lambda(\xi,\eta) = \Lambda(\xi,\eta)\sigma_D(\xi)+\Lambda(\xi,\eta)\sigma_D(\eta).$$

which means that $\sigma_D(\xi)$ is linear in ξ and thus of the form $\langle X,\xi\rangle$ for some fixed vector field X on M. Hence D is of the form $u \to L_X u + au$ for some $a \in N$. It is now an easy job to show that $a \in \mathbb{R}$ and that X is an infinitesimal conformal transformation of the symplectic structure.

4. Let us illustrate another useful way of using the symbol with the following simple example. Let ∇ be a covariant derivative on M and let $\nabla^r : u \to \nabla^r u$ be the differential operator of order r mapping a function u on a r-covariant tensor field defined inductively by $\nabla^{(1)}u = du$ and

$$(\nabla^r u)(X_1,\dots,X_r) = X_1.(\nabla^{r-1}u)(X_2,\dots,X_r) - \sum_{\ell>1}(\nabla^{r-1}u)(X_2,\dots,\nabla_{X_1}X_\ell,\dots,X_r).$$

Then in any local chart of M, the components of $\nabla^r u$ are of the form

$$(\nabla^r u)_{i_1\dots i_r} = \partial_{i_1}\dots\partial_{i_r}u \ + \text{ lower order terms in } u$$

and its total symbol is $\xi \otimes \dots \otimes \xi$. Thus, for a given polynomial $\sigma(\xi_0,\dots,\xi_{p-1})$ of degree $\vec{r}$,

$$\sigma^\nabla(u_0,\dots,u_{p-1}) = \sigma(\nabla^{r_0}u_0,\dots,\nabla^{r_{p-1}}u_{p-1})$$

obtained by contracting each $\nabla^{r_i}u_i$ with the polarization of σ with respect to ξ_i, defines a p-linear differential operator of symbol σ.

For instance, starting with $\sigma = \bar{\sigma}_C$, where C is supposed to be of order $\vec{r}$, we obtain $C^\nabla_{\vec{r}} = \bar{\sigma}^\nabla_C$ and the degree of $C - C^\nabla_{\vec{r}}$ is strictly less than $\vec{r}$. By induction, this gives the canonical decomposition

$$C = \sum_{\vec{s}\leq\vec{r}} C^\nabla_{\vec{s}}$$

of C with respect to ∇. [4]

Cohomology of the Poisson algebra

A. Structure of the symbols of the cocycles

5. We now suppose that M is equipped with a symplectic form F. We denote by μ the canonical extension of the isomorphism $X \in TM \to -i(X)F \in T^{\star}M$ and we set $\Lambda = \mu^{-1}F$. Recall the Poisson bracket defined by

$$\{u,v\} = \Lambda(du,dv) \ , \ u,v \in N,$$

and recall that $(N,\{,\})$ is a Lie algebra : the *Poisson algebra* of (M,F).

We now restrict ourself to skew symmetric p-linear differential operators $C : N^p \to N$ which vanishes on the constants (i.e. $c(u_0,\dots,u_{p-1}) = 0$ if $u_\ell \in \mathbb{R}$ for some $\ell < p$). We call such an operator a p-cochain. Its *coboundary* is by definition the (p+1)-cochain

$$(\partial C)(u_0,\dots,u_p) = \sum_i (-1)^i\{u_i, C(u_0,\dots\hat{i}\dots,u_p)\} + \sum_{i<j} (-1)^{i+j} C(\{u_i,u_j\},u_0,\dots\hat{i}\dots\hat{j}\dots,u_p)$$

where $\hat{i}$ stands for the omission of u_i. If $\partial C = 0$, C is said to be a p-*cocycle* and since $\partial \circ \partial = 0$, one introduces the space

$$H^p_{diff,nc}(N) = \{\text{p-cocycles}\}/\{\text{p-coboundaries}\}$$

which is called the p-th *Chevalley cohomology space of* N for the differentiable cochains vanishing on the constants.

6. It turns out that using the composition formula of §3, one easily computes $\sigma_{\partial C}$ in terms of σ_C and then $\bar{\sigma}_{\partial C}$ in terms of $\bar{\sigma}_C$. The last relation may be written down in the form

$$\bar{\sigma}_{\partial C} = \pm\ \partial_\rho \bar{\sigma}_C$$

where ∂_ρ denotes the coboundary of the Chevalley cohomology of a finite dimensional representation ρ of the symplectic Lie algebra $sp(n,\mathbb{R})$. Such cohomology is completely known [1] according to which the symbols of the cocycles of the Poison algebra can be completely described, as they satisfy $\partial_\rho \bar{\sigma}_C = 0$.

Let P_0 denotes the algebra of polynomials invariant under the natural action of the symplectic group $Sp(n,\mathbb{R})$. Due to this invariance, a $P(\xi_1,\dots,\xi_s) \in P_0$ induces a polynomial on each $T^{\star}_x M$, which will still be denoted by P.

Let also denote by C_0 the algebra of $Sp(n,\mathbb{R})$-invariant scalar forms on $sp(n,\mathbb{R})$. Then

$$\ker \partial_\rho = \operatorname{im} \partial_\rho \oplus (P_0 \otimes C_0).$$

Let T denotes the space of contravariant skew-symmetric tensor fields on M. Then, α denoting the skew-symmetrization :

Theorem. *Adding a coboundary if necessary, the total symbol of a cocycle* C *of total degree* k *may be given the form*

$$\sigma_C = \sum_{|\vec{r}|=k} \alpha(P_{\vec{r}}) \; , \; P_{\vec{r}} \in P_0 \otimes C_0 \otimes T.$$

B. The 3*-differentiable cohomology*

7. Due to the above theorem, it would be interesting to construct *cocycles* having prescribed symbols $\in P_0 \otimes C_0$ because such operators should have some chance to represent classes generating the cohomology. In view of §4, it is easy to construct *cochains* having a given symbol. As we need cocycles, we have to modify that construction. We do not know how to proceed in general but we shall indicate how to work for the elements of P_0 of degree 3 in each argument, the space of which we denote by P_3.

Let Γ be a connection on M and ∇ be its covariant derivative. Define then $\mathcal{L}\Gamma$ by

$$\mathcal{L}_u\Gamma \; : \; (X,Y,Z) \to F((L_{X_u}\nabla)(X,Y),Z)$$

where $X_u = \mu^{-1}(du)$ and $L_\star\nabla$ denotes the Lie derivative of ∇ in the direction of $\star$. If Γ is symplectic (i.e. torsionless and such that $\nabla F = 0$), $\mathcal{L}_u\Gamma$ is a covariant symmetric 3-tensor field on M. As easily seen, given $P \in P_3$,

$$S_\Gamma(P)(u_0,\dots,u_{p-1}) = P(\mathcal{L}_{u_0}\Gamma,\dots,\mathcal{L}_{u_{p-1}}\Gamma)$$

obtained by contracting $\mathcal{L}_{u_i}\Gamma$ with the polarization of P with respect to $\xi_i\,(0 \leq i < p)$, defines *a cocycle of symbol* P *whose cohomology class does not depend on* Γ.

For C_0, the situation is not so clear. There is no cocycles of prescribed symbol in C_0 except for a flat manifold M. However, given Γ, there is a canonical way to construct cochains $\tau_\Gamma(k)$ $(0 \leq k \leq m)$ of order 2 in each argument, whose symbols generate the algebra C_0 and whose coboundaries are of order 1 in each argument.

Let $I_p(N)$ be the space generated by the cochains

$$S_\Gamma(P) \wedge \tau_\Gamma(k_1) \wedge \dots \wedge \tau_\Gamma(k_\ell) \wedge \hat{\Omega}$$

$(P \in \mathcal{P}_3, 0 \leqslant k_1 < \dots < k_\ell \leqslant m\ ,\ \Omega \in \Lambda(M))$, where

$$\hat{\Omega}(u_0, \dots, u_{p-1}) = \Omega(X_{u_0}, \dots, X_{u_{p-1}}) \ ,$$

and observe that $\partial I_\Gamma(N) \subset I_\Gamma(N)$.

8. Let us say that a cochain C is t-*differentiable* if it is of order $\leqslant t$ in each argument, that is if $|\alpha_i| \leqslant t$ for each $i < p$, in the notations of §2. It is easily seen that ∂C is t-differentiable if so does C so that the t-differentiable cohomology $H_{t\text{-diff},nc}(N)$ is defined.

For $t = 1$, it is known to be isomorphic to the de Rham cohomology of M [5].

The computation of $H_{t\text{-diff},nc}(N)$ for $t = 2,3$ follows from the following (observe that the elements of $I_\Gamma(N)$ are 3-differentiable) :

Theorem. *Let* Γ *be a symplectic connection. Then*

$$H_{3\text{-diff},nc}(N) \cong H(I_\Gamma(N), \partial).$$

Using that theorem, one can indeed show that

Theorem. $H_{3\text{-diff},nc}(N) \cong \mathcal{P}_3 \otimes H_{2\text{-diff},nc}$

and that

Theorem. $H_{2\text{-diff},nc}(N) \cong H(\Lambda(H)^{2^m}, \delta)$ *where*

$$\delta(\Omega_{i_1,\dots,i_k}, i_1 < \dots < i_k)$$

$$= ((-1)^k d\Omega_{i_1 \dots i_k} + \sum_j \sum_{i_{j-1} < \ell < i_j} (-1)^{j-1} T_\ell \wedge \Omega_{i_1 \dots i_{j-1} \ell i_j \dots i_k})$$

where T_ℓ *denotes the representative of the* ℓ-th *trace class of* M *obtained by the canonical construction of the Chern-Weil homomorphism of* TM *associated to* Γ.

9. References

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