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Separation of orthogonal sets of measures

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Separation of orthogonal sets of measures

Michel Talagrand (Results of G.Mokobodzki)

Let K be a compact space. Let X be the set of positive measures on X of mass ≤ 1. A subset of X is said to be measure-convex if for each compact set $L \subset A$ and each Radon measure μ on L we have $\int x \, d\mu(x) \in A$.

A function $\varphi: [0,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$ is called a medial limit if it is strongly affine (i.e. universally measurable, and $\varphi(\int x \, d\mu(x)) = \int \varphi(x) \, d\mu(x)$ for each Radon measure μ on $[0,1]^{\mathbb{N}}$) and if for each $x = (x_n) \in [0,1]^{\mathbb{N}}$, $\liminf x_n \leq \varphi(x) \leq \limsup x_n$. Mokobodzki proved that continuum hypothesis implies the existence of a medial limit [1]. (It is known now that Martin's axiom is enough to imply the result).

Theorem (Mokobodzki). Assume there exists a medial limit. Then given two K-analytic measure convex sets $A, B \subset X$ which are orthogonal (i.e. $\mu \in A$, $\nu \in B \Rightarrow \mu \perp \nu$) there exists a universally measurable set $V \subset K$ with $\mu \in A \Rightarrow \mu(V) = 1$, $\mu \in B \Rightarrow \mu(V) = 0$.

Note that, as the result of D.Preiss shows, it is impossible in general to take V Borel.

Proof. We first think to X as a convex compact set of its own, forgetting about its special structure. For any set $A \subset X$, $x \in X$, let

$$\hat{1}_A(x) = \sup \{ \mu^*(A), \delta_x < \mu \}$$

where δ_x is the Dirac measure in x, $\mu \in M^+(X)$, and $\prec$ is the Choquet order, i.e. $\delta_x \prec \mu \Leftrightarrow f(x) \leq \mu(f)$ for each convex continuous f. It is classical that if A is compact

$$\hat{1}_A(x) = \mathrm{Inf}\ \{f(x) ;\ f \text{ affine continuous, } 1_A \leq f\} \quad (1)$$

Hence for each decreasing sequance (A_n) of compact sets,

$$\hat{1}_{\cap A_n}(x) = \inf_n \hat{1}_{A_n}(x) .$$

Now for $A,B \subset X$, let

$$\mathcal{C}'(A,B) = \operatorname*{Sup}_n \hat{1}_A(x) + \hat{1}_B(x) - 1 .$$

If (A_n), (B_n) are two decreasing sequences of compact sets, $\mathcal{C}'(\cap A_n, \cap B_n) = \inf_n \mathcal{C}'(A_n,B_n)$.

For $A,B \subset X$, let

$$\mathcal{C}(A,B) = \mathrm{Inf}\ \{\varepsilon ;\ \exists f \text{ strongly affine on } X \text{ with } 1_A \leq f\ \ 1 - 1_B + \varepsilon\} .$$

Let A_n, B_n be two increasing sequances of sets in X. For each n let $\varepsilon_n \leq 2^{-n} + \mathcal{C}(A_n,B_n)$ such that

$1_{A_n} \leq f_n < 1 - 1_{B_n} + \varepsilon_n$, where f_n is strongly affine.

Let $f(x) = \varphi((f_n(x)))$ where φ is a medial limit (note that $f_n(x) \in [0,1]$, $\forall n, \forall x$). Then f is strongly affine, and $1_A \leq f \leq 1 - 1_B + \varepsilon$, where $\varepsilon = \limsup \varepsilon_n$. This proves that $\mathcal{C}(\cup A_n, \cup B_n) = \sup_n \mathcal{C}(A_n,B_n)$.

Now, suppose A,B compact. If $1_A \leq f < 1 - 1_B + \varepsilon$ where f is strongly affine, from (1) we get $\hat{1}_A \leq f < 1 - \hat{1}_B + \varepsilon$ so $\hat{1}_A + \hat{1}_B - 1 < \varepsilon$. Moreover, if $\hat{1}_A + \hat{1}_B - 1 < \varepsilon$, then $\hat{1}_A \leq 1 - \hat{1}_B + \varepsilon$, and since $\hat{1}_A$ is concave u.s.c., $1 - \hat{1}_B + \varepsilon$ concave ℓ.s.c., the Hahn-Banach theorem shows that there

exists an affine continuous f with , $1_A \leq f < 1 - \hat{1}_B + \varepsilon$.
We have shown that $\mathscr{C}(A,B) = \mathscr{C}'(A,B)$.

We have shown that $\mathscr{C}(A,B)$ is a <u>capacity</u>. Let $A,B \subset X$ as in the statement. The capacitability theorem shows that

$$\mathscr{C}(A,B) = \operatorname{Sup}\{\mathscr{C}(A_1,B_1) , A_1 \subset A, B_1 \subset B, A_1,B_1 \text{ compact}\}.$$

Since A,B are measure convex, we can assume A_1,B_1 convex. Let $x \in X$. It is easy to see that $x = ay + (1-a)y = by' + (1-b)y'$ where $a = 1_{\hat{A}_1}(x)$, $b = 1_{\hat{B}_1}(x)$, $y \in A_1$, $y' \in B_1$.

Now by hypothesis there exists a Borel set $V \subset K$ with $y(V) = 1$, $y'(V) = 0$. Hence we get $1-b \geq x(V) \geq a$ and so $a + b - 1 \leq 0$, that is $\mathscr{C}'(A,B) = \mathscr{C}(A_1,B_1) = 0$. Hence $\mathscr{C}(A,B) = 0$. Using again medial limits, we get a strongly affine f on X with $1_A \leq f \leq 1 - 1_B$. Let g on K given by $g(t) = f(\delta_t)$ for $t \in K$. Since f is strongly affine, for each measure μ on K , $f(\mu) = \int g(t)\, d\mu(t)$. It is clear now the universally measurable set $V = \{t \in K;\ g(t) = 1\}$ works.

[1] P.A.Meyer, Limites mediales, d'après Mokobodski
Seminaire de Probabilités de Strasbourg,1971/72,Springer, Lecture Notes