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In [7], categories of models of infinitary languages $L_{\infty,\infty}(\mu)$ are considered. The main results of this paper show the relations between syntactic properties of theories and semantical properties of their categories of models. The considered infinitary languages are completely general, it means without any smallness condition. The present paper brings the author's results concerning languages subjected to smallness conditions. There will be considered languages $L_{\kappa,\lambda}(\mu)$ admitting conjunctions and disjunctions of less than κ formulas and quantifications over less than λ variables. Here κ and λ are infinite cardinals and μ is a type, i.e. a set of relation and function symbols. The only distinction with respect to [3] is that we admit infinitary relation and function symbols, too. But we shall assume that their arities are smaller than λ. The set of variables of $L_{\kappa,\lambda}(\mu)$ will be denoted by V.

We are going to consider concrete categories Mod(T) of all models of theories T of the just described languages $L_{\kappa,\lambda}(\mu)$. These concrete categories have some immediate properties which we include in our general definition of a concrete category.

Definition: Under a <u>concrete category</u> we shall mean a category $\mathcal{A}$ equipped with a faithful functor $/\ /: \mathcal{A} \longrightarrow$ Set into the category of sets such that the following three conditions are satisfied:

(1) If $A \in \mathcal{A}$, X is a set and $f: /A/ \longrightarrow X$ a bijection then there is $B \in \mathcal{A}$ and an isomorphism $g: A \longrightarrow B$ such that $/B/ = X$ and $/g/ = f$ (the condition of transfer)

(2) If $A, B \in \mathcal{A}$ and $f: A \longrightarrow B$ is an isomorphism such that $/A/ = /B/$ and $/f/ = 1_{/A/}$ then $A = B$ and $f = 1_A$ (the condition of unicity)

(3) $\{A \in \mathcal{A} \ / \ /A/ = X\}$ is a set for any set X (the condition of

fibre-smallness).

In what follows, the symbol //C// will mean the cardinality of the underlying set /C/ of an object $C \in \mathcal{A}$. Let c be a cardinal. We shall take c-filtered colimits in the sense of [5] (where the case $c = \aleph_0$ is considered), i.e. they correspond to c-cofiltered colimits in the sense of [4]. Concerning c-presentable objects and categories see [4].

Let $\mathcal{C}$ be a set of objects of a category $\mathcal{A}$. Let $D: S \longrightarrow \mathcal{A}$. be a c-filtered diagram and $\alpha : D \longrightarrow A$ its colimiting cone in $\mathcal{A}$. We say that α is well-behaved with respect to $\mathcal{C}$ if for any $C \in \mathcal{C}$ and any $f: C \longrightarrow A$ there is $s \in S$ and $g: C \longrightarrow D(s)$ such that $f = \alpha_s . g$.

The comma category $(\mathcal{C} \downarrow A)$ is defined as follows: its objects are couples (C,f) where $f: C \longrightarrow A$ and morphisms $h: (C,f) \longrightarrow (\bar{C},\bar{f})$ are morphisms $h: C \longrightarrow \bar{C}$ in $\mathcal{A}$ such that $\bar{f}.h = f$. Sometimes we shall write $(\mathcal{C} \downarrow_{\mathcal{A}} A)$ in order to stress that the situation is considered in a category $\mathcal{A}$. The assignment

$$(C,f) \longmapsto f \quad , \quad h \longmapsto h$$

gives the projection $P: (\mathcal{C} \downarrow A) \longrightarrow \mathcal{A}$ and the prescription $\alpha_{(C,f)} = f$ gives the canonical cone $\alpha : P \longrightarrow A$. $\mathcal{C}$ is called dense in $\mathcal{A}$ if α is a colimiting cone for any $A \in \mathcal{A}$ (see [5]).

Some results will depend on Löwenheim-Skolem theorem for $L_{\alpha,\lambda}(\mu)$. Owing to the presence of infinitary relation and function symbols, the generalized continuum hypothesis is needed for this theorem. Hence, by writing GCH we shall mean that this hypothesis is assumed.

A theory T is called a strict $\exists!$-theory if it can be given by axioms of the kind

$$(\forall x)(\exists y)(\varphi(x) \rightarrow \psi(x,y)),$$

where $x \in V^n$, $y \in V^m$, n,m are cardinals and φ, ψ are conjunctions of atomic formulas, and such that

$$T \vdash (\forall x)(\forall y)(\forall z)(\varphi(x) \wedge \psi(x,y) \wedge \psi(x,z) \rightarrow y = z)$$

(see [2]).

Theorem 1: Let $\mathcal{A}$ be a concrete category and c a regular cardinal. Then the following conditions are equivalent:

(i) $\mathcal{A} \cong$ Mod(T) for a strict $\exists!$-theory T of $L_{c,c}(\mu)$

(ii) $\mathcal{A}$ is cocomplete, / / is representable by a c-presentable object C_0 and $\mathcal{A}$ contains a dense set of c-presentable objects C admitting an epimorphism $n.C_0 \longrightarrow C$ for some $n < c$.

Corollary: Let $\mathcal{A}$ be a concrete category. Then the following conditions are equivalent:

(i) $\mathcal{A} \cong$ Mod(T) for a strict $\exists!$-theory T of $L_{\aleph,\lambda}(\mu)$

(ii) $\mathcal{A}$ is locally presentable and / / is representable by a presentable object.

A theory T is called <u>coherent</u> if it can be given by axioms of the kind

$$(\forall x)\varphi(x)$$

$$(\forall x)\neg\varphi(x)$$

$$(\forall x)(\varphi(x) \rightarrow \psi(x))$$

where $x \in V^n$, n is a cardinal and φ, ψ are existential-positive formulas (see [6]). Clearly any strict $\exists!$-theory is coherent.

Theorem 2 (GCH): Let $\mathcal{A}$ be a concrete category. Then the following conditions are equivalent:

(i) $\mathcal{A} \cong$ Mod(T) for a coherent theory T of $L_{\aleph,\lambda}(\mu)$

(ii) There is a regular cardinal c such that

a) $\mathcal{A}$ has and / / preserves c-filtered colimits

b) c-filtered colimits are well-behaved with respect to

$\mathcal{C} = \{C \in \mathcal{A} \ / \ //C// < c\}$

c) $\mathcal{C}$ is dense in $\mathcal{A}$ and the comma category $(\mathcal{C} \downarrow A)$ is c-filtered for any $A \in \mathcal{A}$.

A Π_n^0-formula is a prenex formula with a leading quantifier $\forall$ and with n alternations of quantifiers (see [1]). A theory is called a Π_n^0-theory if it can be given by axioms which are Π_n^0-formulas.

Theorem 3 (GCH): Let $\mathcal{A}$ be a concrete category and $n \geq 1$ a natural number. Then the following conditions are equivalent:

(i) $\mathcal{A} \cong Mod(T)$ for a Π_n^0-theory T of $L_{a,\lambda}(\mu)$

(ii) $\mathcal{A}$ contains subcategories

$$\mathcal{A} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \mathcal{M}_n$$

such that the conditions

a) /f/ is injective for any $f \in \mathcal{M}_1$

b) if $f.g \in \mathcal{M}_{k+1}$ and $f \in \mathcal{M}_k$ then $g \in \mathcal{M}_{k+1}$ for any $0 \leq k < n$

are satisfied and there is a regular cardinal c such that

c) $\mathcal{A}$ has colimits of c-filtered diagrams $D: S \longrightarrow \mathcal{M}_n$. Moreover, these colimits are well-behaved with respect to $\mathcal{C} = \{C \in \mathcal{A} \ / \ //C// < c\}$ and they are preserved by / /

d) the comma category $(\mathcal{C} \downarrow_{\mathcal{M}_n} A)$ is c-filtered and A is its canonical colimit for any $A \in \mathcal{A}$.

The last theorem yields a characterization of categories of models of universal-existential theories (i.e. Π_1^0-theories). Remark that any coherent theory is Π_1^0. Having a Π_n^0-theory T, the subcategories $\mathcal{M}_k$, $k > 0$ consist of Σ_{k-}^0 -extensions (in the sense of [1]).

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