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The Mackey topology of a topological vector space $E = (E, \tau)$ is the strongest locally convex topology μ which produces the same continuous linear functionals as the original topology τ of E. If E is an F-space (i.e. metrizable and complete), then μ is easily seen to be the strongest locally convex topology on E which is weaker than τ (cf. [3]). In this case μ is metrizable. The completion $\hat{E}$ of (E, μ) is an F-space which we call the Mackey-completion of E.

N. J. Kalton [2] has shown that the Mackey-completion $\widehat{\ell_\varphi}$ of a separable Orlicz sequence space ℓ_φ is the Orlicz space $\ell_{\hat{\varphi}}$ where $\hat{\varphi}$ is the Orlicz function which coincides with φ on $[1, \infty)$ and is the largest convex function $\leqslant \varphi$ on $[0,1]$.

We present now some results about the Mackey-completions of non-separable Orlicz sequence spaces (cf. [1]).

By an Orlicz function we mean a function $\varphi : [0,\infty) \to [0,\infty)$ which is non-decreasing, strictly positive and left-continuous on $(0,\infty)$, and continuous at 0 with $\varphi(0) = 0$. The Orlicz sequence space ℓ_φ is the vector space of scalar sequences $x = (x_n)$ such that $\sum \varphi(|\varepsilon x_n|) < \infty$ for some $\varepsilon > 0$, with the linear topology λ_φ defined by the F-norm

$$\|x\|_\varphi = \inf\{\varepsilon > 0 : \sum \varphi(|x_n|/\varepsilon) \leqslant \varepsilon\}.$$

The Minkowski functional p_φ of the absolutely convex absorbing subset $k_\varphi = \{x = (x_n) : \sum \varphi(|x_n|) < \infty\}$ of ℓ_φ is a continuous seminorm on ℓ_φ.

We denote by μ_φ, π_φ the Mackey topology on ℓ_φ and the topology defined on ℓ_φ by p_φ, respectively.

Theorem 1:

$$\mu_\varphi = \sup\{\lambda_{\hat{\varphi}}|_{\ell_\varphi}, \pi_\varphi\}.$$

Hence $(\ell_\varphi, \mu_\varphi)$ is normable.

Theorem 2: Let φ, ψ be Orlicz functions. The following conditions are equivalent:

a) $\ell_\varphi \cap \ell_\psi$ is a dense subset of ℓ_ψ.

b) $\ell_\psi \subset \ell_\varphi + k_\psi$.

c) There exist $c > 0$, $d > 0$ and $w_0 > 0$ such that each $w \in [0, w_0]$ can be written as $w = u + v$ $(u, v \geqslant 0)$ so that

$$\varphi(cu) - \psi(2v) \leqslant d\,\psi(w).$$

It is not difficult to prove that if $\psi = \hat{\varphi}$, then the condition c) holds with any $w_0 > 0$, $c = 1$, $d = 3$. Hence we have

<u>Corollary</u>: For every Orlicz function φ, ℓ_φ is a dense subset of $\ell_{\hat{\varphi}}$.

If $\mu_\varphi = \lambda_{\hat{\varphi}}|_{\ell_\varphi}$, then $\widehat{\ell_\varphi}$ may be identified in a natural way with $\ell_{\hat{\varphi}}$.

<u>Theorem 3</u>: The following conditions are equivalent:

a) $\mu_\varphi = \lambda_{\hat{\varphi}}|_{\ell_\varphi}$.

b) There exist $a > 0$, $b > 0$ and $t_0 > 0$ such that

$$\varphi(2t) \leqslant a \max\{\varphi(t), \hat{\varphi}(bt)\} \quad \text{for } t \in [0, t_0].$$

We can construct an Orlicz function which is non-Δ_2, nonequivalent to any convex Orlicz function and yet satisfies b).

Our last result says that if the condition a) fails, then cannot be naturally treated as a sequence space.

<u>Proposition</u>: If $\mu_\varphi \neq \lambda_{\hat{\varphi}}|_{\ell_\varphi}$, then the identity map

$$I : (\ell_\varphi, \mu_\varphi) \longrightarrow \omega$$

does not extend to a continuous linear injection from $\widehat{\ell_\varphi}$ into ω, where ω is the F-space of all scalar sequences.

References

[1] L. Drewnowski and M. Nawrocki, On the Mackey topology of Orlicz sequence spaces, to appear.

[2] N. J. Kalton, Orlicz sequence spaces without local convexity, Math. Proc. Cambridge Philos. Soc. <u>81</u>(1977), 253–277.

[3] J. H. Shapiro, Mackey topologies, representing kernels, and diagonal maps on the Hardy and Bergman spaces, Duke Math. J. <u>43</u>(1976), 187–202.