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Z. Frolík and P. Holický

In [1] we introduced the concepts of σ-discrete decomposability (σ-dd) and hyperanalyticity in uniform spaces. We stated Theorem 1 which clarifies the connection between measurability and discretness, and Theorem 2 which extends the classical (first) separation principle. These results which extend the original Hansell's results for complete metric spaces were supplemented and prepared for publication ([2], [3], [4], [5]).

We choosed an essential part of our results which concern completely regular topological spaces. Since the definitions and terminology are slightly modified, let us start with notations and definitions. All spaces are assumed to be completely regular topological spaces. The letters X,Y stand always for such spaces, the letter $\varkappa$ stands for any infinite cardinal through these notes.

Definitions

(a) A family $\{X_a \mid a \in A\}$ of subsets of X is called σ-dd if there are $X_a^n \subset X_a$ s.t.

$$X_a = \bigcup \{X_a^n \mid n \in \omega\}^{1)}$$

and

$$\{X_a^n \mid a \in A\} \text{ are discrete.}^{2)}$$

(b) A (set-valued) mapping $f : X \to Y$ is called σ-dd-preserving if $\{f[X_a]\}$ is σ-dd in Y whenever $\{X_a\}$ is σ-dd

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1) $\omega = \{0,1,2,\dots\}$

2) discrete = discrete in some continuous pseudometric through this note !

in X .

(c) X is called $\varkappa$-analytic if there is a compact-valued σ-dd-preserving mapping f from a complete metric space M of weight less or equal $\varkappa$ onto X s.t.

$$f^{-1}[C] = \{m \in M \mid f(m) \cap C \neq \emptyset\}$$

is closed in M whenever C is closed in X . Such triples (f,M,X) are called $\varkappa$-analytic parametrizations of X .

(d) The elements of the smallest σ-algebra closed under discrete unions of cardinality $\varkappa$ and containing zero sets of continuous real functions are called $\varkappa$-Baire sets.

Theorems 1 and 2 of [1] are still valid if we write $\varkappa$-analytic or $\varkappa$-Baire instead of hyperanalytic or hyper-Baire, respectively.

We are going to formulate several results concerning characterizations of $\varkappa$-analytic spaces, images of measurable sets, and measurable selections.

Characterizations

Theorem 1. The following statements concerning X are equivalent:

(a) X is $\varkappa$-analytic.

(b) X is a Suslin subset of K×M where K is compact and M is a complete metric space of weight $\varkappa$.

(c) X is a paracompact Suslin subset of a Čech complete space of weight $\varkappa$.

(d) There is a $\varkappa$-analytic parametrization $(f, \varkappa^{\omega}, X)$ where $\varkappa$ is (a set of cardinality $\varkappa$) endowed with discrete metric.

The condition (d) has an auxiliary character for obtaining other characterizations and results, and unables to extend the classical methods which are based upon the connection between ω-analytic spaces and irrational numbers (ω^ω) .

Using essentially Theorems 1 and 2 from [1] we derived results concerning

Images of measurable sets

Theorem 2. Let X be a $\mathcal{H}$-analytic space, P a metric space, and $f : X \longrightarrow P$ a σ-dd-preserving mapping s.t. $f^{-1}[S]$ is Suslin in X whenever S is Suslin in P . Then:

(a) $f^{-1}[B]$ is $\mathcal{H}$-Baire in X if B is Borel (= Baire) in P.

(b) The graph of f , i.e. the set

$$\{(x,p)\in X\times P \mid p\in f(x)\} ,$$

is $\mathcal{H}$-analytic in $X\times P$.

(c) The set $f[X]$ is $\mathcal{H}$-analytic in P .

(d) If $f^{-1}[B]$ is a $\mathcal{H}$-Baire set in X then B is $\mathcal{H}$-Baire in $f[X]$.

The statement (d) says that f is " $\mathcal{H}$-Baire-quotient" (c.f. [6]).

The following characterization of "bimeasurable" mappings between complete metric spaces and Theorem 2 gives a good picture of measurable images of Borel (= Baire) sets in the case of complete metric spaces.

Theorem 3. Let M,P be complete metric spaces of weight $\leq \mathcal{H}$ and let $f : M \longrightarrow P$ be a ω-Baire measurable σ-dd-preserving mapping. Then f is $\mathcal{H}$-Baire measurable, and f^{-1} is $\mathcal{H}$-Baire measurable iff the family $\{\{p\} \mid f^{-1}(p)$ is uncountable$\}$ is σ-dd (i.e. its union is "σ-discrete").

The sufficiency of the last condition can be proved similarly as the well-known analogy in separable spaces (N. Luzin) using Theorems 1 and 2 from [1] , and the "second separation principle"

([7] , Th. 14).

The other implication could be reduced to the separable case in the following way: We find an analogy to a Mazurkiewicz--Sierpiński theorem [8] , Th. 2, p. 506 , and prove that

$$U = \{p \mid f^{-1}(p) \text{ is uncountable}\}$$

is $\mathcal{K}$-analytic. If it were not σ-discrete we find an uncountable compact subset $K \subset U$. Using Theorem 2 from [1] again we show that $f^{-1}[K]$ is a separable Borel subset of M , and we can use Purves's result ([9] , Theorem).

An other object of our investigation are

Selections

We prove analogies to von Neumann ([10] , Lemma 16) and Mazurkiewicz ([8] , Th. 3, p. 491) theorems:

Lemma. Let P be a closed subspace of $\mathcal{K}^{\omega}$ (c.f. Theorem 1(d)), and let h be a continuous σ-dd-preserving mapping of P onto X . Consider the selection $s : X \to P$ s.t. $s(x)$ is the smallest (in lexicographical order) element of $h^{-1}(x) \subset \mathcal{K}^{\omega}$. Then

(a) $s^{-1}[G] \in (\mathcal{S} \setminus \mathcal{S})\sigma$-d [3]

(b) $P \setminus s[X]$ is Suslin in P .

The following selection theorem can be easily derived:

Theorem 4. Let $F : X \to Y$ be a set-valued mapping, and let the graph of F be point-analytic. [4] Let the projection of graph

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3) $\mathcal{S} \setminus \mathcal{S}$ stands for differences of Suslin subsets of X ; $(\mathcal{S} \setminus \mathcal{S})\sigma$-d denotes σ-discrete unions of elements of $\mathcal{S} \setminus \mathcal{S}$

4) There is an analytic parametrization (f,M, graph F) where f is a single-valued correspondence

F into X is σ-dd-preserving. Then F admits a selection s.t. $F^{-1}[G] \in (\mathcal{S} \setminus \mathcal{S})\sigma$-d for open G, and if graph F is point-Luzin 5) then the selection f can be chosen s.t. moreover, graph $F \setminus$ graph f is Suslin.

Theorem 1 from [1] implies that compact-valued Suslin-measurable mapping from an analytic space into a metric space fulfills the assumptions of Theorem 4.

The method of Kuratowski and Ryll-Nardzewski [11] can be also used for derivation of selection theorems. For an example see [12].

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5) There is an analytic parametrization (f,M, graph F) where f is a one-to-one mapping

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