WSAA 8

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1. Basic concepts

Let $\varrho : R^n \times R^n \longrightarrow R$ be a metric on R^n having following properties:

1) ϱ is invariant under translations, i.e.

$$\varrho(x,y) = \varrho(x+z,\ y+z) \quad \text{for all} \quad x,y,z \in R^n$$

2) ϱ is linear euklidian, i.e.

$\varrho(x,y) : \varrho(x',y') = \varrho_e(x,y) : \varrho_e(x',y')$ holds for all points x,y,x',y' belonging to a straight line, where ϱ_e denotes the euklidian distance in R^n.

(It is obvious that every metric on R^n fulfilling 1) and 2) induces a norm $\| \ \|_\varrho$ on R^n).

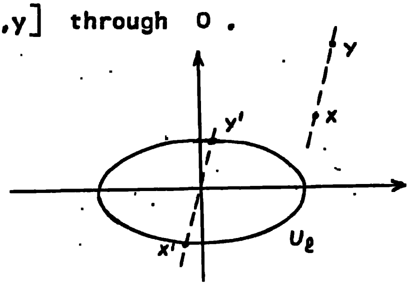
For such metrization on R^n we obtain

$$(1) \qquad \varrho(x,y) = 2\,\frac{\varrho_e(x,y)}{\varrho_e(x',y')}$$

where $[x',y']$ is the diameter of the unit ball

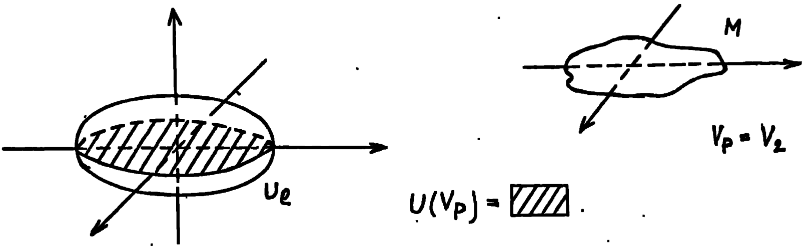
$$U_\varrho = \{u \in R^n / \varrho(0,u) \leq 1\}$$

parallel to $[x,y]$ through 0.



The formula (1) is the starting point in the paper of H. Busemann [1].

Let M be a Lebesgue measurable subset of R^n and V_p a p-flat containing M $(1 \leq p \leq n)$. We denote by $U(V_p)$ the set in which the p-flat parallel to V_p through O intersects the unit ball U.



In analogy to property 2) for ϱ Busemann desired for a p-dimensional measure m^p

$$m^p(M) : m^p(U(V_p)) = \lambda^p(M) : \lambda^p(U(V_p))$$

and he defined

$$m^p(U(V_p)) = \lambda^p(U^p_{\varrho_e}) ,$$

where λ^p denotes the p-dimensional Lebesgue measure and $\lambda^p(U^p_{\varrho_e})$ the p-dimensional Lebesgue measure of the p-dimensional euclidian unit ball $U^p_{\varrho_e}$. This means

$$(2) \qquad m^p(M) = \frac{\lambda^p(U^p_{\varrho_e})}{\lambda^p(U(V_p))} \lambda^p(M) .$$

m^p is called the p-dimensional <u>Busemann measure</u> of M (with respect to ϱ). It is clear, that $m^p(M)$ depends on the position of M in (R^n, ϱ). m^p is invariant under translation but in general it is not invariant under rotation.

2. Relation between Busemann measures and Hausdorff measures

Let us denote by h^p the p-dimensional Hausdorff measure, i.e.

$$h^p(M) = \sup_{\varepsilon > 0} \inf \left\{ \sum_{i=1}^{\infty} \sigma^p(A_i) / \bigcup_{i=1}^{\infty} A_i \supset M \wedge \sigma(A_i) \leq \varepsilon \right\} .$$

For p-dimensional Hausdorff measures holds:

Theorem (Rogers [2]). If $M \subset R^n$ is a Lebesgue measurable subset of the n-dimensional euclidian space $(R^n, \| \|_e)$, then

$$h^n_{\| \|_e}(M) = \frac{1}{\lambda^n(U_{\| \|_e})} \lambda^n(M) .$$

A conclusion of this theorem is the following:
If $M \subset R^n$ is a Lebesgue measurable subset of R^n and $\| \|$ an arbitrary norm on R^n, then

$$h^n_{\| \|}(M) = \frac{1}{\lambda^n(U_{\| \|})} \lambda^n(M) .$$

Let us now consider a p-flat V_p in (R^n, ϱ) through $0 \in R^n$ $(1 \leq p \leq n)$. The set

$$U(V_p) = V_p \cap U_\varrho$$

induces a norm $\| \|_{V_p}$ having $U(V_p)$ as unit ball. By the above conclusion, it holds for $(V_p, \| \|_{V_p})$:

$$h^p_{\| \|_{V_p}}(M) = \frac{1}{\lambda^p(U(V_p))} \lambda^p(M) .$$

Now we obtain the following

Theorem. If M is a Lebesgue measurable subset of $(R^n, \| \|)$ and V_p a p-flat containing M, then

$$m^p(M) = \lambda^p(U^p_{\varrho_e}) \, h^p_{\| \|_{V_p}}(M) .$$

The proof is given by the equality above and the equality (2).
It is easy to prove that

$$h^p_{\| \|}(M) = h^p_{\| \|_{V_p}}(M)$$

for every subset M of $(R^n, \| \|)$. This implies together with the Theorem above.

Theorem. For every Lebesgue measurable subset M of $(R^n, \| \|)$

holds

$$m^p(M) = \lambda^p(U^p_{\ell_e})\, h^p_{\|\ \|}(M) .$$

References

[1] H. Busemann: The Foundations of Minkowskian Geometry Comm.Math.Hev. 24 (1950), 156-187

[2] C.A. Rogers: Hausdorff Measures, Cambridge 1970