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ON SETS ALWAYS OF THE FIRST CATEGORY

by

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We obtain topological analogies of some results from [2] and [3] concerning measures. We work in ZFC set theory.

$|S|$ denotes the cardinality of a set S. If f is a function from a set S into a set T and $\mathcal{F}$ is a family of subsets of S then by $f(\mathcal{F})$ we denote the family $\{f(F): F \in \mathcal{F}\}$ of subsets of T. Let $\mathcal{C}$ be a separable σ-field on S (i.e. $\mathcal{C}$ is a countably additive algebra of subsets of S such that $\mathcal{C}$ is countably generated and $\{s\} \in \mathcal{C}$ for every $s \in S$). We will write $\mathcal{C} \in N$ iff there is no continuous probability on $\mathcal{C}$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are σ-fields on S then $\sigma(\mathcal{C}_1, \mathcal{C}_2)$ denotes the σ-field on S generated by $\mathcal{C}_1 \cup \mathcal{C}_2$. We will write $\mathcal{C} \in Q$ iff there is no metrizable separable without isolated points topology $\mathcal{T}$ on S such that $\mathcal{B}(\mathcal{T}) \supset \mathcal{C}$ and $S \notin I(\mathcal{T})$, where $\mathcal{B}(\mathcal{T})$ is the usual Borel σ-field on S w.r.t. (with respect to) $\mathcal{T}$ and $I(\mathcal{T})$ denotes the σ-ideal of the first category subsets of S w.r.t. $\mathcal{T}$. By $\mathcal{B}$ we will denote the usual Borel σ-field on R (= the real line) w.r.t. the usual topology on R.

REMARK. If in the above definition of the class Q we will replace $I(\mathcal{T})$ by $I^*(\mathcal{T})$, where $I^*(\mathcal{T})$ denotes the σ-ideal of subsets of S which are always of the first category w.r.t. $\mathcal{T}$, then we obtain the same class Q. It can be also observed that if $X \subset S$, $\mathcal{C}$ is a separable σ-field on S and $\mathcal{C} \in \mathcal{Q}$ then $\mathcal{C} \cap X$ is a separable σ-field on X such that $\mathcal{C} \in \mathcal{Q}$.

Recall that a subset X of S is always of the first category w.r.t. metrizable separable topology $\mathcal{T}$ on S iff each dense in itself subset Y of X is of the first category on itself w.r.t. the topology on Y induced by $\mathcal{T}$.

It is known the following

THEOREM 1 ([3]). There exist a separable σ-field $\mathcal{C}$ on a set S such that $\mathcal{C} \notin \mathcal{N}$ and a permutation φ of S such that $\sigma(\mathcal{C}, \varphi(\mathcal{C})) \in \mathcal{N}$.
If all subsets of R of cardinality $< 2^{\aleph_0}$ are Lebesgue measurable then we can additionally have $S = R$ and $\mathcal{C} = \mathcal{B}$.

We give sketch of the proof of the following topological analogue of Theorem 1.

THEOREM 2. There exist a separable σ-field $\mathcal{C}$ on a set S such that $\mathcal{C} \notin \mathcal{Q}$ and a permutation φ of S such that $\sigma(\mathcal{C}, \varphi(\mathcal{C})) \in \mathcal{Q}$.
If all metrizable separable spaces without isolated points of cardinality $< 2^{\aleph_0}$ are of the first category then we can additionally have $S = R$ and $\mathcal{C} = \mathcal{B}$.

Proof. Our proof is a modification of Prikry's method from [4]. Let $m = \min \{ n$: there exists a separable σ-field

$\mathcal{C}$ on a set T such that $|T| = n$ and $\mathcal{C} \notin Q\}$. Hence there exists a set T with $|T| = m$ and a metrizable separable without isolated points topology $\mathcal{T}_0$ on T such that $T \notin I(\mathcal{T}_0)$.

It can be easily observed the following

LEMMA. Let $\mathcal{T}$ be a metrizable separable without isolated points topology on a set T such that $|T| = m$. Then $T' \subset T$ and $|T'| < m$ imply $T' \in I(\mathcal{T})$.

Let $\{t_\xi : \xi \in m\}$ be an one-to-one enumeration of T. Let for every $\xi \in m$, F_ξ be an F_σ (w.r.t. $\mathcal{T}$) subset of T such that $F_\xi \in I(\mathcal{T})$ and $F_\xi \supset \{t_{\xi'} : \xi' \leqslant \xi\}$. Put $Z = \bigcup_{\xi \in m} \{t_\xi\} \times F_\xi$. We have $Z \subset m \times T$. It can be proved (compare [4], [1] and [2]) that there exists a separable σ-field $\mathcal{C}$ on m such that Z belongs to the product σ-field $\mathcal{C} \otimes \mathcal{B}(\mathcal{T})$. We claim $\mathcal{C} \in Q$. If not, let $\mathcal{T}_1$ be a metrizable separable without isolated points topology on m such that $\mathcal{B}(\mathcal{T}_1) \supset \mathcal{C}$ and $m \notin I(\mathcal{T}_1)$. Applying Kuratowski-Ulam category version of Fubini's theorem and our Lemma to $\mathcal{T}_0$, $\mathcal{T}_1$ and Z one can easily obtain a contradiction (compare [4] and [3]). So $\mathcal{C} \in Q$.

Let S be a set such that $|S| = m$ and let $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$ and $|S_1| = |S_2|$. Since $|S_1| = m$ and $|S_2| = m$ it follows that there exist a separable σ-field $\mathcal{C}_1$ or S_1 such that $\mathcal{C}_1 \in Q$ and a separable σ-field $\mathcal{C}_2$ on S_2 such that $\mathcal{C}_2 \notin Q$. Let $\mathcal{C}$ be the σ-field on S generated by $\mathcal{C}_1 \cup \mathcal{C}_2$. Since $\mathcal{C} \cap S_2 = \mathcal{C}_2$ and $\mathcal{C}_2 \notin Q$ we have, by Remark, $\mathcal{C} \notin Q$. Let φ be a permutation of S

such that $\varphi(S_1) = S_2$. It can be chacked that our $\mathcal{C}$ and φ have the required properties.

We omit the easy proof of the additional claim of Theorem 2.

If $\mathcal{J}$ is a σ-ideal on R then we put $\mathcal{J}^+ = \{A \subset R:$ for every $B \subset R$ such that there exists 1 – 1 $\mathcal{B}$-measurable function $f: B \longrightarrow A$ we have $B \in \mathcal{J}\}$.

Observe that $\mathcal{J}^+$ is a σ-ideal on R and $\mathcal{J}^+ \subset \mathcal{J}$.

Denote by $\mathcal{L}_0$ the σ-ideal of subsets of R of the Lebesgue measure zero, by I the σ-ideal of the first category subsets of R w.r.t. usual topology on R and by I^* the σ-ideal of subsets of R which are always of the first category w.r.t. usual topology on R.

Let $m_0 = \min\{n: \exists(Y \subset R)(|Y| = n \text{ and } Y \notin \mathcal{L}_0\}$,

$m_1 = \min\{n: \exists(Y \subset R)(|Y| = n \text{ and } Y \notin I\}$, and

$m_2 = \min\{n: \exists(Y \subset R)(|Y| = n \text{ and } Y \notin I^*)\}$.

We omit the proof of the following

THEOREM 3. There exist $A_i \subset R$ $(i \leqslant 3)$ such that $|A_i| = m_i$ $(i \leqslant 2)$, $|A_3| = \min\{m_0, m_1\}$, $A_0 \in \mathcal{L}_0^+$, $A_1 \in I^+$, $A_2 \in (I^*)^+$ and $A_3 \in \mathcal{L}_0^+ \cap I^+$.

If is known that the σ-ideal $\mathcal{L}_0^+$ is equal to the σ-ideal of so called universal null subsets of R [5].

The part of Theorem 3 concerning A_0 was proved in fact in [2].

COROLLARY. There exists $A, B \subset R$ such that $|A| = |B|$, $A \in I^*$ and $B \notin I^*$.

A similar result for universal null subsets of R can be found in [2]. In connection with I^* it is worth to mention that Morgan II has proved [3a] that there exists a linear set every homeomorphic image of which is in I, but which is not in I^*.

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