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SOME TOPOLOGICAL ASPECTS OF THE THEORY OF TOPOLOGICAL TRANSFORMATION GROUPS

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There are very much interesting and deep results in the classic case, when the subjects for research are topological transformation groups with metrizable compact phase spaces and Lie action groups. The list of all topologists working in this area is very long: A. Borel [1], H. Cartan [2], C. Chevalley [3], A. Gleason [4], J.L. Koszul [5], D. Montgomery [6], G.D. Mostow [7], R. Palais [8], C.T. Yang [9], L. Zippin [10] and others. But I suppose that the general case is waiting yet for its serious studying.

Here we'll speak about some new results of Soviet topologists S. Bogaty [11], M. Madirimov [12], B. Pasynkov [14] and Yu.M. Smirnov [13] in the following directions: DIMENSION, EXTENSION, EMBEDDING for general topological transformation groups.

0. The general transformation group or shortly the G-space is by definition a topological space X (phase space) with a given continuous action $\alpha : G \times X \longrightarrow X$ of a given topological group G (action group). It is convenient to write g(x) instead of $\alpha(g,x)$. A subset A of a given G-space X is called invariant, if $g(A) \subset A$ for every element g of G. A continuous mapping $f: X \longrightarrow Y$ of G-spaces X and Y is called equivariant, if $f(g(x)) = g(f(x))$ for every element g of G and every point x of X.

1. EXTENSION. Let K be a class of topological spaces and G be a fixed group. Then we denote by K(G) the class of all G-spaces for which its phase spaces belong to K. The question is following: Is it possible to characterize the property of a given G-space Y to be an absolu te (neighborhood) extensor for a given class K(G) in terms of pure topological properties of some subsets Y!H! of Y ?

Here, naturally, a G-space Y is called an absolute (neighborhood) extensor, briefly AE (resp. ANE) for a given class K(G), if for every G-space X of K(G) and every closed invariant subset A of X, every equivariant mapping $f: A \longrightarrow Y$ has an equivariant extension $F: X \longrightarrow Y$ (resp. $F: U \longrightarrow Y$, where U is an invariant neighborhood of

A in X).

Some necessary conditions are investigated for the following two cases: i) in the "compact case" when the action group G is compact and the class K consists of all compact spaces, ii) in the "metrizable case", when the group G is metrizable and the class K consists of all metrizable spaces. Namely we have the

Theorem 1 S (Smirnov [13 a]). If in these two cases a G-space Y is AE (ANE) for the class K(G) then for every closed subgroup H of G the set Y!H! is AE (resp. ANE) for the class K. Here Y!H! is the set of all "H-fixed" points, i.e. $Y!H! = \{ y \in Y \mid h(y) = y \quad \forall\ h \in H \}$.

This theorem is true also under the following supplementary condition:

AE(n) (resp. ANE(n)): If we take in the definition of the property AE (resp. ANE) for the given topological class K the extensionable mappings $f: A \longrightarrow Y!H!$, where $A \subset X$, with the property $\dim (X \setminus A) \leqq n + 1$ (for a fixed number n), then we shall take in the definition of the property AE (resp. ANE) for the given "G-class" K(G) the extensionable mappings $f: A \longrightarrow Y$, where $A \subset X$, with the property $\dim (X \setminus A) \leqq n + 1 + \dim G$.

Problem 1. Is this theorem true for the "PP-case", when the action group G is a P-paracompact space (in the sense of Arhangelski [15]) and K consists of all P-paracompact spaces ? and for "paracompact case" ??

Problem 2. For which groups G is true the proposition inverse to the proposition of Theorem 1 S ? Is it for compact metrizable zero-dimensional groups ?? Is it for compact metrizable groups ??? and so on ...

The answer is positive for finite commutative action groups, as has been proved by Bogaty:

Theorem 1 B (Bogaty [11 a]). Let the action group G be finite and commutative. Then in metrizable case a finite dimensional G-space Y is AE (resp. ANE) for the class K(G) if and only if for every closed subgroup H of G the set Y!H! is AE (resp. ANE) for the class K.

This theorem is true also under the supplementary condition AE(n) (resp. ANE(n)) without a given restriction of finite dimensionality on the space Y. Then one puts in the proposition of Theorem 1B that the sets Y!H! are LC^n (resp. C^n & LC^n).

Jan Jaworowski [16] has proved this theorem only for the cyclic group Z_p of a prime order p and M. Madirimov [12 a] has established this result for every cyclic group Z_n. The method used by Bogaty-Madirimov is different from the Jaworowski's method.

2. DIMENSION. There are well known theorems in the dimension theory about some characteristics of the dimension dim X for metrizable spaces X given by M. Katětov [17] and K. Morita [18]. The question is following: Which of these characteristics have some "equivariant" generalizations?

Some of such "equivariant" characteristics of dimension are investigated by M. Madirimov for metrizable phase spaces X and finite action groups G in the following manner. The family ω of sets of a given G-space X is called equivariant, if $g(U) \subset U$ for every U of ω and every g of G (Jaworowski [16 a]).

Theorem 1 M (Madirimov [12 b]). For every n-dimensional metrizable G-space X with finite action group G

i) there exist some zero-dimensional metrizable G-space M and some equivariant closed mapping h: M ⟶ X onto X, such that the inverse image $g^{-1}(x)$ consists of no more than n + 1 points for every point x of X,

ii) there exist n + 1 zero-dimensional invariant sets $U_o, \ldots, U_n$ such that $X = U_o \cup \ldots \cup U_n$,

iii) there exists an open basis ω such that $\dim(\mathrm{Bd}\ U) \leqq n - 1$ for every set U of ω and ω is a sum of countably many locally-finite equivariant systems ω_i.

Naturally, all inverse propositions are true too by the corresponding topological theorems given by M. Katětov [17] and K. Morita [18].

Corollary. For the orbit space X/G of every metrizable G-space X with a finite action group G the dimension dim X/G = dim X.

Problem 3. For which action groups are true the propositions and Corollary of Theorem 1 M ? Is it for compact metrizable zero-dimensional groups?? Is it for metrizable zero-dimensional groups ??? and so on ...

Problem 4. Is the equality dim X = dim G + dim X/G true for locally-compact phase space X and locally compact action group G, such that all orbits are closed in X ? And if only for compact case??

3. EMBEDDING. Here we have two natural questions: the question about existence of universal G-spaces for some classes of G-spaces and the question about the linearization of some actions on some G-spaces. In this direction there are many interesting papers of P.S. Baayen [19], D.H. Carlson [20], J. de Groot [21], G.D. Mostow [7], R.S. Palais [8 b], J. de Vries [22 ab] and other authors. The last results about universal G-spaces were received I suppose by J. de Vries [22 b] and myself [13 b] independently and almost simultaneously.

Let L(X) be the Lindelöf degree of a topological space X (i.e. the minimum of cardinal numbers n such that every open covering of X has a subcovering of cardinality n).

Theorem 1 V (de Vries [22 b]). For every infinite locally compact group G there exists the G-space U(G) with completely regular phase space such that every completely regular G-space X of weight w(X), where $w(X) \leqq L(G)$, can be topologically and equivariantly embedded into the universal G-space U(G).

It is very interesting his second [22 b]

Theorem 2 V. The G-space X of weight w(X), where $w(X) \leqq L(G)$, is topologically and equivariantly embeddable in some compact G-space Y if and only if the G-space X is bounded (see below). If G is σ-compact and X is separable and metrizable, then Y may be supposed to be metrizable too.

Here the G-space X is said to be bounded, if it is bounded with respect to some uniformity Ω on X (i.e. for every entourage O of Ω there exists a neighborhood U of identity element e of G such that from $g(x) \in U \times G$ follows $(g(x),x) \in O$). +)

I was studying some functorial dependence $\mathfrak{X}$ between the maps $h: X \longrightarrow Y$, where Y is a G-space with an action α, and maps $\mathfrak{X}(h)$: $: X \longrightarrow Y^G$, where Y^G is the space of all maps (continuous or not) from G to Y with some natural action β_G and compact-open topology x).

+) See also the report "Embeddings of G-spaces" given by J. de Vries at this Symposium (Part B of the Proceedings).

x) All "functional" spaces are taken here and below with compact-open topology only!

The "functor" $\tilde{\alpha}$ preserves the following properties of maps h: i) injectivity, ii) continuity, iii) equivariance, if X is a G-space too, iv) a property to be a topological embedding, v) a property to be a closed topological embedding, if the action group G is compact. Here by definition, a topological embedding h is closed, if the image h(X) is closed. In the case ii) $h_\alpha(X) \subset C(G,Y)$, where $h_\alpha = \tilde{\alpha}(h)$ and C(G,Y) is the set of all continuous mappings from G to Y.

The "functor" $\tilde{\alpha}$ is some topological embedding of the space C(X,Y) into the space C(X,C(G,Y)) always supposing the G-space X and the topological space Y are fixed. If Y is a given topological vector space, then the "functor" $\tilde{\alpha}$ is a monomorphism. If G is a locally compact group, then the restriction of the action β_G to $G \times C(G,Y)$ is a continuous action on the group G on the space C(G,Y). Consequently in this case the map $h_\alpha = \tilde{\alpha}(h)$ is an equivariant continuous mapping to C(G,Y) for every continuous mapping h. For any locally convex space Y there exists a continuous monomorphism $\mu_G : G \longrightarrow L(Z)$, where L(Z) is the group of all topological linear transformations of the space Z = C(G,Y).

<u>Theorem 2 S</u> (<u>Smirnov [13 b]</u>). <u>Every completely regular</u> G<u>-space</u> X <u>with a locally compact action group</u> G <u>has some topological equivariant embedding</u> h_α <u>into some locally convex space</u> Z <u>with the natural action of the group</u> G <u>as subgroup of the group</u> L(Z) (<u>then this space</u> Z <u>is some universal</u> G<u>-space). If the action group</u> G <u>is compact, then the embedding</u> h_α <u>may be supposed to be closed</u>.

The proof of this theorem is different from the proof of Theorem 1 V given by J. de Vries. I think that the weight condition given by J. de Vries in Theorems 1 V and 2 V can be received in our way too. Almost all our propositions of Theorem 2 S can be illustrated by the following commutative diagram:

$$\begin{array}{ccccc}
G & \times & X & \xrightarrow{\ \alpha\ } & X \\
\Big\downarrow & \mathrm{Id}\times h_\alpha & \Big\downarrow & & \Big\downarrow h_\alpha \\
G & \times & Z & \xrightarrow{\ \beta\ } & Z \\
\Big\downarrow & \mu\times\mathrm{Id} & \Big\downarrow & \nearrow \nu & \\
L(Z) & \times & Z & &
\end{array}$$

Here Z = C(G,Y) and $\nu(p,z) = p(z)$ naturally.

Theorem 2 S is a corollary to one embedding theorem given by R. Arens [23] and the following

Theorem 3 S. Every continuous mapping (resp. topological embedding or closed topological embedding) $h: X \longrightarrow Y$ has some corresponding continuous mapping (resp. topological embedding or closed topological embedding always supposing the action group G is compact) $h_\alpha : X \longrightarrow Z$ in some space Z ($= C(G,Y)$) such that for some action $\beta = \beta_G$ on Z and some monomorphism $\mu = \mu_G$ all conditions of Theorem 2 S are satisfied.

The "program" of our proof is the following: we define the maps h_α and $\tilde{\alpha}$, the action β and the monomorphism μ by the formulae $h_\alpha(x)(g) = h(\alpha(g,x))$, $\beta(g',f)(g) = f(g,g')$, $\mu(g)(f) = \beta(g,f)$.

Theorem 4 S. The map $\tilde{\tilde{\alpha}}$ by the hypothesis of Theorem 3 S is a topological isomorphic embedding.

Problem 5. Are Theorems 2 S, 3 S and 4 S true without the condition of local compactness of the action group G ?

4. DIMENSION. Some dimension theorems for topological groups given by B. Pasynkov [14 a] are generalized recently by the same author. Let $\pi : X \longrightarrow X/G$ be the natural projection of a given G-space X on its orbit space X/G. B. Pasynkov calls a completely regular G-space X with the compact action group G almost metrizable if the projection π is a perfect mapping and the orbit space X/G is metrizable.

Theorem 1 P (Pasynkov [14 b]). If the G-space X is almost metrizable, then $\dim X = \operatorname{Ind} X = \Delta X$, where ΔX is the dimension in the sense of V. Ponomarev [24].

Theorem 2 P. Every finite dimensional almost metrizable G-space X has some zero-dimensional perfect mapping on some metric space M, with $\dim M \leqq \dim X$.

To these results is closely related the following

Theorem 3 P. Every almost metrizable compact G-space is dyadic and, moreover, is some Dugundji space [25].

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