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RECENT DEVELOPMENT OF THEORY OF  
UNIFORM SPACES

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Since the 4th Prague Symposium there has been a fast development of the theory of uniform spaces. Many old problems have been solved, and the solutions have opened new areas. Here I want to report just on two subjects: some sort of non-separable descriptive theory (cozero-sets, Baire sets etc.) and measure theory. My interest in uniform spaces originated in these two subjects. The preliminary introduction to uniform methods in descriptive theory was given in my talk on the 4th Prague Symposium and more definite program was introduced in my talks on conferences in Budva [ $F_2$ , 1972], Athens (Ohio 1972) and Pittsburgh [ $F_3$ , 1972]. A great help for me was the work of A.W. Hager; he was primarily interested in lattices of functions, however, he introduced to uniform spaces one of the most useful constructions (described in § 1 as  $\mathfrak{C} - c$ ), and in the separable case observed that hereditarily metric-fine implies that the cozero sets form a  $\mathcal{C}$ -algebra. Also the work of F. Hansell on  $\mathcal{C}$ -discrete decomposability of disjoint completely Suslin-additive families in complete metric spaces (proved now by P. Holický and the present author for products of complete metric spaces by compact spaces) was basic for considering the program.

As concerns the measure theory on uniform spaces, introduced by D.A. Rajkov and independently by L. LeCam, one should consult [ $F_4$ ],[ $F_5$ ],[ $F_6$ ] and recent papers by J. Pachtl where one can find complete bibliography.

In 1973 I started a special seminar on uniform spaces which was attended from the very beginning by several graduate students who put a lot of will-power and enthusiasm in the work of the seminar. The results are published in seminar notes SUS 73-4, SUS 74-5 (very informal), and SUS 75-6 where one can find the details of all results presented here, and also references. In particular, SUS 73-4 contains a long survey of the material connected with cozero, Baire

etc. sets and functions.

Uniform spaces theory could have always been attractive because of its conceptual value for topology, geometry and analysis. Unfortunately, difficult technical procedures were quite exceptional, and the answers to questions coming from the other fields seemed to be just formal reformulations which did not focus on the problems. I feel that these days the theory of uniform spaces has its deep parts, and what is important, is giving the answers with "comments" which may be useful.

Let me just mention several groups of problems which are open for further investigation:

A. Uniform isomorphism problem for locally convex linear spaces: are any two uniformly isomorphic spaces isomorphic as topological linear spaces. It seems that no contra-example is known, however a number of results in positive direction has been given (Unflo, Mankiewicz, Vilímovský).

B. The complexity of uniform covers in general, in particular, the complexity of uniform covers of Banach spaces. The first non-trivial result is due to M. Zahradník, SUS 73-4, who showed that the uniform structure of no infinite-dimensional normed space is generated by  $\mathcal{L}_1$ -continuous partitions of unity. J. Pelant has developed a theory of combinatorial complexity of uniform covers using new technical tools, and the recent examples of spaces without point-finite basis due to V. Růdl and J. Pelant are very simple, however they are using more involved combinatorics (it should be mentioned that the first examples are due to Ščepin and Pelant).

C. Lattices and ordered sets of uniformities. Many results on atoms have been published by J. Pelant, J. Reiterman and P. Simon. In uniform spaces these topics are more complicated than in topological spaces, however ultrafilters play an important role.

D. The category of uniform spaces is quite interesting from the categorial point of view. Here I want just to mention that one is forced to use functors to be able to say something non-trivial and reasonable about various concrete problems; e.g. note the role of the plus and minus functors in what follows. Two classical facts are used frequently (Isbell, for the second also [Čl]):

(1) every uniform space can be embedded into a product of complete metric spaces which can be chosen injective;

(2) every uniform space is a quotient of a "devil space"  $D(\mathcal{F})$ :  $\mathcal{F}$  is a filter, say on a set  $X$ , the underlying set of  $D(\mathcal{F})$  is

$Z \times X$ , and the uniformity of  $D(\mathcal{F})$  has the following covers  $\mathcal{U}_F$ ,  $F \in \mathcal{F}$ , for a basis;  $\mathcal{U}_F$  consists of all singletons  $\langle 0, x \rangle$  and  $\langle 1, x \rangle$  for  $x \notin F$ , and all two-point sets  $\{\langle 0, x \rangle, \langle 1, x \rangle\}$  for  $x \in F$ .

It should be noted that the usage of the devil space was developed by M. Hušek.

E. Spaces of uniformly continuous mappings into topological linear spaces, in particular, into the reals. Two problems have been studied, existence of extensions and stability of the spaces with respect to the pointwise defined algebraic operations. In the real valued case J. Pelant, J. Vilímovský and the present author showed that the two problems are connected, and they described the largest coreflective class with the extension property. The general case has been studied by J. Vilímovský who showed that the situation is more complicated; the definite answers are not known. It should be noted that many results are implicitly contained in the literature on geometry of Banach spaces.

Now we are coming to the proper subject of my talk. The results will be stated in the terminology described in § 1.

### § 1. Notation and basic constructions.

We start with a description of three constructions which will be used frequently in describing various functors.

A. If  $\mathcal{C}$  is a coreflective class of uniform spaces (closed under sums and quotients, and, of course, isomorphisms) then the class  $\text{sub } \mathcal{C}$  consisting of subspaces of spaces in  $\mathcal{C}$  is coreflective, and if  $c$  is the coreflection on  $\mathcal{C}$  then  $\text{sub}c$  stands for the coreflection on  $\text{sub } \mathcal{C}$ . Moreover, the functor  $\text{sub}c$  is evaluated as follows: if  $X \hookrightarrow Y$ , and if  $Y$  is injective, then  $\text{sub}c X \hookrightarrow \text{sub}c Y$ . This general result is formulated by J. Vilímovský, the method was invented by J. Isbell who used it in the case of topologically fine spaces (the method was also used by M. Rice for metric- $t_p$  spaces).

Remark. It is natural to ask the following question: Under what conditions the product of two spaces in a coreflective class  $\mathcal{C}$  belongs to  $\mathcal{C}$ ? The problem was opened by J. Isbell and Poljakov for the case of proximally fine spaces, and reconsidered by V. Kůrková by showing "yes" if one of the spaces is compact (or more generally, precompact). Her work was followed by nice results of M. Hušek, recently jointly with M. Rice, in general setting.

B. If  $\mathcal{C}$  is a coreflective class then the class  $\text{her } \mathcal{C}$  of all

$X$  such that each subspace of  $X$  belongs to  $\mathcal{C}$  does not need to be coreflective (e.g. take (A.D.) Alexandrov spaces for  $\mathcal{C}$ ). However, if it is, then here  $c$  denotes the corresponding coreflection provided that  $c$  is the coreflection on  $\mathcal{C}$ .

C.  $\mathcal{K} - c$  spaces. Let  $\mathcal{K}$  be a class of spaces, and let  $c$  be a coreflection. The class  $\mathcal{K} - c$  consists of all  $Y$  such that if  $X \in \mathcal{K}$ , and  $f: Y \rightarrow X$  is uniformly continuous, then so is  $f: Y \rightarrow cX$ . This construction was introduced to uniform spaces by A. Hager.

The rest of this § is devoted to constructs associated with the concept of "refinement" introduced explicitly in [F<sub>3</sub>]. If  $\mathcal{K}$  is a category (always concrete) denote by  $\text{Set}_{\mathcal{K}}$  the category on objects of  $\mathcal{K}$  such that  $\text{Set}_{\mathcal{K}}(X, Y)$  is the set of all mappings of  $X$  into  $Y$ . Here  $\mathcal{K}$  is usually the category  $U$  of uniform spaces. A refinement of  $\mathcal{K}$  is any category between  $\mathcal{K}$  and  $\text{Set}_{\mathcal{K}}$ . The following sequence of refinements of  $U$  will be used frequently:

$$U \hookrightarrow \mathcal{D} \hookrightarrow p \hookrightarrow \text{coz} \hookrightarrow t \hookrightarrow \text{Set}_U.$$

Recall [F] that  $\mathcal{D}(X, Y)$  is the set of all distal mappings from  $X$  into  $Y$  (the preimages of discrete collections are discrete),  $p(X, Y)$  is the set of all proximally continuous mappings from  $X$  into  $Y$ ,  $\text{coz}(X, Y)$  is the set of all coz-mappings of  $X$  into  $Y$  (preimages of coz-sets are coz-sets, the coz-sets are preimages of open sets under the uniformly continuous functions), and  $t(X, Y)$  is the set of all continuous functions of  $X$  into  $Y$ .

It should be remarked that every concrete functor of  $\mathcal{K}$  into any category defines a refinement, and every refinement is generated in this way.

Indeed, if  $L: \mathcal{K} \rightarrow \mathcal{L}$  is a concrete functor define a refinement  $\mathcal{R}_L$  as follows:

$$\mathcal{R}_L(X, Y) = \mathcal{L}(LX, LY).$$

Usually we write  $L$  for  $\mathcal{R}_L$ . For example,  $t$  is the usual functor of  $U$  into topological spaces, which assigns to each uniform space the induced topological space. Similarly for the functor  $p$  of  $U$  into proximal spaces. The refinement  $\text{coz}$  may be defined by the functor  $\text{coz}$  into paved spaces:  $\text{coz } X$  is the set  $X$  endowed with the collection of all coz-sets in  $X$ . On the other hand, if  $\mathcal{L}$  is a refinement of  $\mathcal{K}$ , then  $\mathcal{L}$  is generated by the obvious functor of  $\mathcal{K}$  into the category  $\langle \mathcal{K} \rangle_{\mathcal{L}}$  defined as follows: the objects are the equivalence classes under the relation:  $X$  and  $Y$  are isomorphic in

$\mathcal{L}$  under the identity mapping of  $X$  onto  $Y$ ; the equivalence class containing  $X$  is denoted by  $\langle X \rangle_{\mathcal{L}}$ .

Remark. In general, the approach to the refinements by means of functors into other categories has many advantages, e.g. simplicity. On the other hand, there are natural refinements "from life", e.g. the refinement in the § on measure theory.

Consider a refinement  $U \hookrightarrow \mathcal{L}$ . Denote by  $\text{Inv}(\mathcal{L})$  (more precisely,  $\text{Inv}(U \hookrightarrow \mathcal{L})$ ) the class of all concrete functors  $F: U \rightarrow U$  such that  $X$  and  $FX$  are isomorphic in  $\mathcal{L}$  by the identity mapping; these functors are called  $\mathcal{L}$ -preserving. Denote by  $\text{Inv}_+(\mathcal{L})$  or  $\text{Inv}_-(\mathcal{L})$  the class of all positive or negative functors in  $\text{Inv}(\mathcal{L})$ , accordingly. Recall that a functor  $F$  is called positive (negative) if the identity mapping  $X \rightarrow FX$  ( $FX \rightarrow X$ ) is uniformly continuous for each  $X$ . Note that concrete reflections are just the idempotent positive functors, and similarly for coreflections.

If there exists the coarsest functor in  $\text{Inv}_+(\mathcal{L})$ , it is called the plus-functor of  $\mathcal{L}$  and denoted by  $\mathcal{L}_+$ . Similarly,  $\mathcal{L}_-$  is defined. We say that  $\mathcal{L}_+$  is strong if  $\mathcal{L}_+$  is the coarsest element in  $\text{Inv}(\mathcal{L})$ . Self-evidently the plus and minus functors are idempotent.

Before discussing the properties of  $+$  and  $-$  functors, let me recall the concepts of fine and coarse objects. An object  $X$  is called  $\mathcal{L}$ -fine if

$$U(X, Y) = \mathcal{L}(X, Y)$$

for each  $Y$ . If the relation is satisfied for all  $X$ , then  $Y$  is called  $\mathcal{L}$ -coarse.

The class of all  $\mathcal{L}$ -fine objects is coreflective (easy), and the corresponding coreflection is denoted by  $\mathcal{L}_f$ . Similarly, the reflection on  $\mathcal{L}$ -coarse objects is denoted by  $\mathcal{L}_c$ . It is easy to show that the following three conditions are equivalent:

1.  $\mathcal{L}_f = \mathcal{L}_-$ .
2.  $\mathcal{L}_f$  preserves  $\mathcal{L}$  (i.e.  $\mathcal{L}_f \in \text{Inv}(\mathcal{L})$ ).
3.  $\mathcal{L}_-$  generates  $\mathcal{L}$  (i.e.  $\mathcal{L}(X, Y) = U(\mathcal{L}_-X, \mathcal{L}_-Y)$ ).

If the three conditions are satisfied then  $\mathcal{L}$  is called fine (this is equivalent to the statement that  $U$  is reflective in  $\mathcal{L}$ ). Similarly for "coarse".

Classical simple results say that  $p$  is coarse, and  $t$  is fine. It is easy to show that  $t_+ = p$  (strong), and it follows from the

fact that  $p(X \times X)$  uniquely determines  $X$ , that  $p_-$  is the identity (strong).

It is interesting that sub  $p_f$  is the identity (Hušek, Vilímovský), and in November 1976 J. Pelant (with certain help of P. Pták) showed that sub  $t_f$  is the locally fine coreflection, solving an old problem of J. Isbell.

It is shown in  $[F_3]$  that  $\mathcal{D}$  is coarse. Note that  $\mathcal{D}$  and related larger refinements have been studied by P. Pták.

The rest is devoted to several refinements related to descriptive theory and measure theory.

## § 2. Refinement $\text{coz}_-$ .

Simple examples show that  $\text{coz}$  is neither coarse nor fine. It is easy to show that  $\text{coz}_c$  is the reflection on the indiscrete spaces. For many results on  $\text{coz}_f$  we refer to SUS 73-4, and SUS 74-5. Here I want just indicate the results on  $+$  and  $-$  functors. If  $\mathcal{L}$  is a refinement, denote by  $\mathcal{L}^2$  the refinement consisting of all  $f: X \rightarrow Y$  such that  $f \times f: X \times X \rightarrow Y \times Y$  is in  $\mathcal{L}$ .

Theorem.  $\text{coz}_- = (\text{coz}^2)_f = \text{metric} - t_f$ . The functor  $\text{coz}_-$  is evaluated at  $X$  as follows:

a. The  $\text{coz}$ -sets in  $X \times X$  containing the diagonal form a basis for the vicinities of the diagonal of  $\text{coz}_X$ .

b.  $\mathcal{G}$ -discrete completely  $\text{coz}$ -additive covers of  $X$  form a basis for the uniform covers of  $\text{coz}_X$ .

There is no reasonable description of the morphisms in  $\text{coz}_-$  except for the one in Theorem, the obvious conjectures fail to be true. It should be remarked that M. Rice found independently a description of  $\text{metric} - t_f$  similar to that in (b). The present author characterized  $\text{metric} - t_f$  spaces by several other properties ( $\ell_\infty$ -partitions are  $\ell_1$ , uniformly continuous maps into metric spaces are preserved by taking continuous limits). It seems that  $\text{coz}_-$  is one of the most useful functors.

It is obvious that  $\text{coz}_X = \text{coz}_f X$  iff  $\text{coz}_X$  is proximally fine. By general method (M. Rice, see A in Introduction), or directly one can show

$$\text{sub } \text{coz}_- = (\text{complete metric}) - t_f,$$

and it follows from the factorization theorem of G. Tashjian ( $\text{coz}$ -mappings of products into metric spaces factorize through countable sub-products) that

$$\text{sub coz}_- = \text{sub coz}_F.$$

Very useful is the coreflection  $\text{her coz}_-$ , which is called the measurable coreflection. A space  $X$  is in  $\text{her coz}_-$  iff it is in  $\text{coz}_-$  and  $\text{coz } X$  is a  $\sigma$ -algebra iff uniformly continuous mappings into metric spaces are closed under taking of pointwise limits of sequences.

It is easy to show that  $\text{coz}_+ = p$ , and it can be proved that  $(\text{coz}_-)_+ = \mathcal{D}$ .

### § 3. Refinement $h \text{ coz}$ .

The hyper-coz sets in a uniform space are the elements of the smallest collection of sets which contains all coz-sets, and it is closed under taking  $\sigma$ -discrete unions. The hyper-coz mappings are defined obviously. The properties of the resulting refinement  $h \text{ coz}$  are similar to those of  $\text{coz}$ , however the proofs are more involved. Clearly

$$U \hookrightarrow h \text{ coz} \hookrightarrow t,$$

and neither  $\text{coz} \subset h \text{ coz}$  nor  $h \text{ coz} \subset \text{coz}$ .

Theorem.  $h \text{ coz}_- = (h \text{ coz}^2)_F = \text{coz}_- \circ \lambda = \text{coz}_- \circ \text{sub } t_F$ . In addition,  $h \text{ coz}_-$  is evaluated at  $X$  as follows: the hyper-coz sets in  $X$  containing the diagonal form a basis for the vicinities of the diagonal.

Corollary.  $\text{sub } (h \text{ coz}_-) = \text{sub } t_F$ .

It follows from Theorem that to evaluate  $h \text{ coz}_-$  at  $X$  it is enough to know  $\text{coz } \lambda X = h \text{ coz } X$ , and  $\mathcal{D}_c \circ \lambda X$ . The distal structure of  $\lambda X$  may be much finer than that of  $X$ . Therefore another functor is of certain interest, namely (metric  $\times$  compact) -  $t_F$ . This coreflection is evaluated as in (b) in Theorem in § 2 with  $\text{coz}$  replaced by  $h \text{ coz}$ .

It can be proved that

$$h \text{ coz}_+ = (h \text{ coz}_-)_+ = \mathcal{D}.$$

### § 4. Refinements $\text{Ba}$ and $h \text{ Ba}$ .

The Baire sets in  $X$  are the elements of the smallest  $\sigma$ -algebra containing the coz-sets. The hyper-Baire sets in  $X$  are the elements of the smallest  $\sigma$ -algebra containing the coz-sets, and closed under discrete unions. The set of all Baire-measurable mappings of  $X$  into  $Y$ , called simply Baire mappings, is denoted by  $\text{Ba}(X, Y)$ . Similarly  $h \text{ Ba}(X, Y)$  stands for the set of all hyper-Baire mappings



of  $X$  into  $Y$ . The properties of the resulting refinements  $Ba$  and  $hBa$  depend on the model of set theory used, and the absolute results I know require quite deep properties of Suslin sets. Obviously

$$U \hookrightarrow \text{coz} \hookrightarrow Ba \hookrightarrow \text{Set}_U,$$

$$U \hookrightarrow h \text{coz} \hookrightarrow hBa \hookrightarrow \text{Set}_U,$$

and it is easy to see that  $Ba$  and  $hBa$  are unrelated. Also  $Ba_+ = p$ , and  $hBa_+ = \mathcal{D}$ . Certainly, none of the two refinements is coarse or fine.

It can be shown that  $Ba_-$  and  $hBa_-$  exist, however no description is known, and we shall see that the evaluation of the two functors at metric spaces depends on the model of set theory used. The absolute results are:

A.  $Ba_p X = Ba_- X = \text{her } \text{coz}_- X$

if  $X$  is a complete metric space.

B.  $hBa_p X = hBa_- X = \text{her } (\text{compact} \times \text{complete metric} - t_p)$

if  $X$  is the product of a compact space by a complete metric space.

It is easy to show that A holds for all separable metric spaces under CH, and A does not hold for  $Q \subset \mathbb{R}$  (an uncountable subset of the reals such that each subset of  $Q$  is a  $G_\delta$ , in particular,  $Ba Q$  is the power set of  $Q$ ). J. Fleisner has announced a result which implies that A is consistent for all metric spaces of cardinal  $\leq \omega_1$ , and A holds for all metric spaces in a model.

For the proof of the absolute results one needs to know the following Lemma which was proved recently by P. Holický and the present author:

if  $X$  is a hyper-analytic space, in particular, if  $X$  is the product of a complete metric space by a compact space, then every disjoint completely Suslin-additive family in  $X$  is  $\sigma$ -discretely decomposable.

It should be remarked that for A one needs the case when  $X$  is complete metric, and this case is due to F. Hansell. Recall that  $\{X_a\}$  is  $\sigma$ -discretely decomposable if there exists a family  $\{X_{a,n}\}$  such that each  $\{X_{a,n}\}_a$  is discrete, and  $X_a = \bigcup \{X_{a,n} \mid n \in \omega\}$ .

For A one needs also a Lemma due to D. Preiss: every disjoint completely Baire-additive family is of a bounded class.

The statements of the categorial consequences are left to the

of  $\mathcal{M}_U(X)$  into  $E$ . Clearly  $\mathcal{M}_U(X)$  is uniquely determined up to an isomorphism preserving  $\sigma^X$ . Clearly every  $f \in U(X, Y)$  extends uniquely to a continuous linear map  $\mathcal{M}_U(f)$  of  $\mathcal{M}_U(X)$  into  $\mathcal{M}_U(Y)$ , thus  $\mathcal{M}_U$  is a functor. The elements of  $\mathcal{M}_U(X)$  are called uniform measures on  $X$ .

It is not difficult to show that  $\mathcal{M}_U(X)$  can be identified with the set of all  $\mu \in \mathcal{M}(X)$  which are continuous in the pointwise topology on each UEB set, endowed with the topology of uniform convergence on UEB sets. The embedding  $\sigma^X$  assigns to each  $x \in X$  the evaluation at  $x$ , i.e. the Dirac measure at  $x$ . One can show that the set  $Mol(X)$  of all molecular measures, i.e. the linear space generated by Dirac measures, is dense in  $\mathcal{M}_U(X)$ . Thus  $\mathcal{M}_U(X)$  is a completion of  $Mol(X)$  endowed by the topology of uniform convergence on UEB sets. The topology of  $\mathcal{M}_U(X)$  is called the uniform topology. The linear space  $U_b(X)$  is the dual of  $\mathcal{M}_U(X)$ , and what is important for our purposes, on the positive cone  $\mathcal{M}_U^+(X)$  the uniform topology coincides with the weak topology ( $= \sigma(\mathcal{M}_U(X), U_b(X))$ ).

The concept of uniform measure may be useful for measure theory because uniform measures are preserved by projective limits,  $\sigma$ -additive measures defined on  $\sigma$ -algebras and also cylindrical measures can be viewed as uniform measures. One can define vector valued uniform measures, and develop a nice theory of integration; the basic result for this purpose is a recent theorem of J. Pachl which says that relatively weakly compact subsets of  $\mathcal{M}_U(X)$  are relatively compact. It is natural to ask whether some questions about uniform measures can be reduced to consideration of uniform measures on very simple uniform spaces. One problem of this sort is considered here.

What can be said about negative functors  $F$  of uniform spaces such that the middle vertical arrow in the diagram

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{M}_U^+(X) & \hookrightarrow & \mathcal{M}_U(X) \\ \uparrow & & \uparrow & & \uparrow \\ FX & \hookrightarrow & \mathcal{M}_U^+(FX) & \hookrightarrow & \mathcal{M}_U(FX) \end{array}$$

is a homeomorphism. The answer is that there exists the finest one, and the refinement  $\mathcal{M}$  of  $U$  generated by the finest one can be described as follows:

$f \in \mathcal{M}(X, Y)$ , iff  $Mol^+(f)$  extends to a continuous mapping of  $\mathcal{M}_U^+(X)$  into  $\mathcal{M}_U^+(Y)$ .

Of course,  $Mol^+(f)$  is the restriction to  $Mol^+(X)$  of the extension  $Mol(f)$  of  $f$  to a linear mapping of  $Mol(X)$  into  $Mol(Y)$ .

I don't know any direct proof. In my proof one considers at the same time a functor by means of playing with "true" Radon measures, and working with both  $\mathcal{M}$  and the functor constructed one finally shows the proposition about  $\mathcal{M}$ , and proves that the functor is  $\mathcal{M}_f$ . The construction of  $\mathcal{M}_f$  is based on the following description of uniform measures [F<sub>5</sub>]:

a measure  $\mu \in \mathcal{M}(X)$  is uniform iff  $\check{\mu}$  is sitting on  $K(\mathcal{U})$  as a Radon measure for each uniform cover  $\mathcal{U}$  of  $X$ .

Here  $K(\mathcal{U})$  is the union of closures in  $\check{X}$  of the elements of  $\mathcal{U}$ .

Remark. This description implies that on a complete metric space the uniform measures are just the Radon measures.

The description of  $\mathcal{M}_f$ :

$\mathcal{M}_f$  is  $X$  endowed with the coarsest uniformity such that all the identity maps

$$\mathcal{M}_f X \rightarrow t_f U$$

are uniformly continuous, where  $U$  runs over all  $U \subset \check{X}$  such that for each uniform measure  $\mu$  on  $X$  the measure  $\check{\mu}$  is sitting on  $U$  as a Radon measure.

The spaces  $\mathcal{M}_f X$  have good properties. They are locally fine (but  $\mathcal{M}_f \neq \mathcal{R}$ ), and  $\mathcal{G}$ -additivity of all uniform measures on  $X$  is described by simple properties of  $X$  as follows.

Theorem. The following properties of  $X$  are equivalent:

1. Each uniform measure on  $X$  is  $\mathcal{G}$ -additive (i.e.  $f_n \downarrow 0$ ,  $f_n \in U_b(X) \implies \mu(f_n) \rightarrow 0$ ).
2.  $\mathcal{M}_f X$  is metric -  $t_f$  (i.e.  $\text{coz}_- \mathcal{M}_f X = \mathcal{M}_f X$ ).
3.  $\mathcal{M}_f X$  is inversion-closed (i.e. if  $f > 0$  is a uniformly continuous function then so is  $1/f$ ).
4. If  $f_n \downarrow 0$ , and  $\{f_n\} \subset U_b(\mathcal{M}_f X)$ , then  $\{f_n\}$  is UEB.

Remark. The properties 3. and 4. are always equivalent (M. Zahradník, SUS 73-4).

The details of § 6 will appear in SUS 76-7, see also [F<sub>6</sub>].

It is an open problem whether there exists a non-trivial positive functor  $F$  such that the middle vertical arrow in an analogous diagram is a homomorphism (even a bijection).

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