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by

M. Rajagopalan

ABSTRACT.

We introduce a set theoretic axiom $\clubsuit_\infty$ which is weaker than $\clubsuit$ as well as axiom F. Using (CH) and $\clubsuit_\infty$ we prove the existence of a locally compact, T_2, locally countable, first countable, hereditarily separable, sequentially compact non-compact space X. The one point compactification X^* of X is a compact, T_2, C-space (meaning X^* is of countable tightness) which is not sequential. We also construct a compact, T_2, C-space Y which is not sequential using only the continuum hypothesis (CH). This solves some well known problems on S-spaces and also on compact C-spaces under least set theoretic axioms.

INTRODUCTION.

Some areas of current interest in topology are cardinal functions and the role of set theoretic axioms. Much literature has grown around these topics. (See [1,2,4,5,7,8,9,10,11]). Set theoretic axioms are used mainly to construct examples like S-spaces. An S-space is a hereditarily separable completely regular space which is not Lindelöf. Spaces which come close to being an S-space are C-spaces in the sense of Mrowka and Moore [6] or, in the language of cardinal functions, spaces X whose tightness t(X) is countable. If Y is a space, the tightness t(Y) of Y is the least among the cardinals λ with the property that if $A \subset Y$ and $x_o \in \bar{A}$ then there is a subset $B \subset A$ of cardinality λ so that $x_o \in \bar{B}$. A C-space is a space Y whose tightness $t(Y) \leq \aleph_o$. The sequential spaces are C-spaces. A space S is called sequential if given $A \subset X$ we can get $\bar{A}$ by iterating the operation of taking limits of convergent sequences beginning from A. We give a more elaborate definition of sequential spaces below. The following problem has been raised several times by A.V. Arhangelskii and also by V. Kannan [5] and Ponomorov [11]. The problem is:

"IS A COMPACT, T_2, C-SPACE SEQUENTIAL?"

The first ones to raise a related problem are S. Mrowka and C.C. Moore [6] who asked whether a Hausdorff C-space is sequential. An example of a Hausdorff C-space which is not sequential was given by Franklin and Rajagopalan and they raised the problem whether a regular, C-space must be sequential. (See [3]).

The above problem of Kannan and Arhangelskii on compact, T_2, C-spaces can be answered in the negative by assuming strong set theoretic axioms. Thus using continuum hypothesis (CH) and the axiom $\clubsuit$ Ostazewski [7] constructed a locally compact, T_2, sequentially compact, first countable, locally countable, hereditarily separable non-compact space X. Such a space was also constructed by Fedorchuk in [2] using axiom F which is stronger than both (CH) and $\clubsuit$.

So the hard question is "what are the least set of axioms which guarantee the existence of such S-spaces as the ones constructed by Ostazewski or which guarantee the existence of compact, T_2, C-spaces which are not sequential?"

In this paper we introduce an axiom $\clubsuit_\infty$ which is weaker than $\clubsuit$. We show that (CH) and $\clubsuit_\infty$ together imply the existence of an S-space such as the one got by Ostazewski. We also show that assuming (CH) alone; there is a compact, T_2, C-space which is not sequential.

NOTATIONS.

We consider only Hausdorff spaces. We assume ZFC which is Zermelo-Frankael set theoretic aximos with axiom of choice. If we use axioms beyond ZFC in set theory in any of our lemmas or theorems we mention only those axioms in the hypothesis of those lemmas or theorems. We follow [12] for basic notions in topology. N is the set of integers $> o$ with discrete topology and βN is its Stone-Čech compactification. Ω is the first uncountable ordinal. If A,B are sets then A/B is the set difference A - B. (CH) denotes the continuum hypothesis. We follow [10] for statements of the axioms (CH), $\diamondsuit$, $\clubsuit$, (MA) and $\circledast$. If X is a topological space and π a partition of X then X/π denotes the quotient space of X given by π. The axiom (F) is stated in Fedorchuk [2].

DEFINITION 1.

Let X be a topological space. Let $A \subset X$. A is called sequentially open if no sequence lying in X/A converges to an element of A. X is called sequential if and only if every sequentially open subset A of X is open. X is called a C-space if given $A \subset X$ and an element $x_o \in \bar{A}$ there is a countable subset $B \subset A$ so that $x_o \in \bar{B}$.

AXIOM $\clubsuit_n$.

Let n be a given integer $> o$. The axiom $\clubsuit_n$ is the following: For every limit ordinal α in $[1, \Omega)$ there are n sets $A_{\alpha 1}$,

$A_{\alpha 2}, \ldots A_{\alpha n}$ so that the following hold:

(a) $A_{\alpha i} \subset [1,\alpha)$ for $i=1,2,..,n$.

(b) $A_{\alpha i}$ is cofinal with $[1,\alpha)$ for all $i=1,2,\ldots,n$.

(c) Given an uncountable subset $M \subset [1, \Omega)$ there exists $\alpha < \Omega$ and $i \in \{1,2,\ldots,n\}$ so that $M \supset A_{\alpha i}$.

AXIOM $\clubsuit_F$.

This is the following statement. Given a limit ordinal $\alpha \in [1, \Omega)$ there is an integer n_α and sets $A_{\alpha 1}, A_{\alpha 2}, \ldots, A_{\alpha n_\alpha}$ satisfying the following:

(i) $A_{\alpha i} \subset [1,\alpha)$ and is cofinal with $[1,\alpha)$ for all α in $[1, \Omega)$ and $i=1,2,..,n_\alpha$.

(ii) Given an uncountable subset $B \subset [1, \Omega)$ there is an $\alpha \in [1, \Omega)$ and an 'i' so that $1 \leq i \leq n_\alpha$ such that $A_{\alpha i} \subset B$.

AXIOM $\clubsuit_\infty$.

This is the following statement. Given a limit ordinal α in $[1, \Omega)$ there exist sets $A_{\alpha 1}, A_{\alpha 2}, \ldots, A_{\alpha n}, \ldots$ so that the following hold:

(I) $A_{\alpha n} \subset [1,\alpha)$ and is cofinal with α for all $n \in N$.

(II) Given an uncountable subset $B \subset [1, \Omega)$ there is an ordinal $\alpha \in [1, \Omega)$ and an integer $n \in N$ so that $A_{\alpha n} \subset B$.

We notice that the axiom $\clubsuit$ of Ostazewski is our $\clubsuit_1$. Clearly $\clubsuit$ implies $\clubsuit_n$ and $\clubsuit_n$ implies $\clubsuit_F$ for all $n \in N$. Moreover $\clubsuit_F$ is easily seen to imply $\clubsuit_\infty$. It is natural to ask whether any of these implications is reversible. But we do not go into it here. The axiom (F) implies $\clubsuit$.

We proceed to prove the following two theorems here:

THEOREM I.

(CH) + $\clubsuit_\infty$ imply that there exists a locally compact, T_2, first countable, hereditarily separable, sequentially compact, locally countable non-compact space S. The one point compactification S^* of S is a compact, T_2, C-space which is not sequential.

THEOREM II.

(CH) alone implies the existence of a compact, T_2, C-space which is not sequential.

We begin to prove Theorem I. We use V-process. The V-process is described in [10]. We begin with the following lemma:

DEFINITION 2.

Let $\underline{A}$ be a countable collection of closed sets of βN and Y an open dense subset of βN such that $\bigcup_{X \varepsilon \underline{A}} X \subset Y$. $\underline{A}$ is called a discrete collection in Y if for every subcollection $\underline{B}$ of $\underline{A}$ we have that $\bigcup_{X \varepsilon \underline{B}} X$ is closed in Y.

DEFINITION 3.

If $\alpha \varepsilon [1, \Omega]$ then λ_α denotes the α^{th} limit ordinal in $[1, \Omega]$. In other words $\lambda_\alpha = \omega^\alpha$ for all $\alpha \varepsilon [1, \Omega]$ where ω is the least limit ordinal in $[1, \Omega]$.

Now we will follow the V-process method of [10] with a slight alteration. For this we will define a closed non-empty subset B_γ of βN for each ordinal γ in $[1, \Omega]$ so that $B_\gamma \cap B_\delta = \phi$ if $\gamma \neq \delta$ and $\gamma, \delta < \Omega$ and $\bigcup_{\gamma<\alpha} B_\gamma$ is a dense open subset of βN for all limit ordinals $\alpha < \Omega$. Then the collection $\{B_\gamma | \gamma \varepsilon [1, \Omega]\}$ will give a partition π of $Y_{\Omega-} = \bigcup_{\gamma<\Omega} B_\gamma$ and $Y_{\Omega-}/\pi$ will be the required locally compact space of Theorem I.

DEFINITION 4.

For every $n \varepsilon N$ we put $B_n = \{n\}$. Put $\underline{A_1}$ to be the collection of all infinite subsets of $\{B_n | n \varepsilon N\}$. $\underline{A_1}$ is well ordered as $A_{11}, A_{12}, \ldots, A_{1\delta}, \ldots$ using (CH) where $\delta \varepsilon [1, \Omega]$. $Y_1 = N = \bigcup_{n<\omega} B_n$ and $\pi_1 = \{\{n\} | n \varepsilon N\}$ by definition.

LEMMA 5.

Let Y be a dense open subset of βN and π a partition of Y by compact sets of βN. Let $\{n\} \varepsilon \pi$ for $n \varepsilon N$. Let the following be satisfied:

(a) Y/π is locally compact and T_2.

(b) Given $A \varepsilon \pi$ there is a compact, open subset V of βN so that V is a countable union of members of π and $A \subset V$.

Let $\underline{A_n} = \{P_{n1}, P_{n2}, \ldots, P_{nr}, \ldots\}$ be a countably infinite collection of members of π so that $\bigcup_{n=1}^{\infty} \underline{A_n} = \{P_{ij} | i,j \varepsilon N\}$ is a discrete collection in Y (see Definition 2). Then there are non-empty compact subsets $C_1, C_2, \ldots, C_n, \ldots$ of βN so that the following hold:

(i) $Y \cup C_n$ is open in βN for all $n \in N$.

(ii) $C_n \neq \phi$ and $C_n \cap C_m = \phi$ for all $n, m \in N$ so that $n \neq m$.

(iii) Given $n \in N$ there is a compact open set V_n of βN so that $C_n \subset V_n$ and V_n/C_n can be expressed as a countable union of members of π.

(iv) Given $n, b \in N$ and a compact open subset W of βN containing C_n we have that $W \supset P_{ki}$ for some $i \in N$.

PROOF.

Let $N_1, N_2, \ldots, N_k, \ldots$ be a pairwise disjoint collection of infinite subsets of N so that $\bigcup_{k=1}^{\infty} N_k = N$. Given $n, r \in N$ find a compact open subset V_{nr} of βN so that $P_{nr} \subset V_{nr} \subset \beta N$ and V_{nr} is a union of countably many members of π and $V_{nr} \cap V_{ms} = \phi$ if either $n \neq m$ or $r \neq s$ for all $n, m, r, s \in N$. Such a family V_{nr} is easily seen to exist by the hypothesis ⓐ and the discreteness of $\bigcup_{n=1}^{\infty} A_n$. Let $W_i = \bigcup_{n=1}^{\infty} (\bigcup_{r \in N_i} V_{nr})$ for all $i \in N$. Let $C_n = \overline{W}_n - W_n$ for all $n \in N$. Then, this family $C_1, C_2, \ldots, C_n, \ldots$ is the required family of sets.

DEFINITION 6.

Let $A \subset \beta N$. The growth A^* of A is defined as $\overline{A}/A$.

LEMMA 7.

Let $Y \subset \beta N$ be a dense open set and π a partition of Y by compact sets so that Y/π is locally compact, T_2 and countable. Assume further that given a member $A \in \pi$ there is a compact open set W of βN so that $A \subset W \subset Y$ and W is a union of members of π. Let $A_1, A_2, \ldots, A_n, \ldots$ be a sequence of distinct members of π so that the growth A^* of the set $A = \bigcup_{n=1}^{\infty} A_n$ is non-empty and disjoint with Y. Then there is a dense open set M of βN with the following properties:

(a) M/Y is a non-empty compact open set.

(b) If $\pi_o = \pi \cup \{M/Y\}$ then π_o is a partition of M so that M/π_o is a countable, locally compact, T_2, space.

(c) There is a compact open set W of βN so that $(M/Y) \subset W \subset M$ and $W \cap Y$ is a union of members of π.

(d) $A^* \cap M \neq \phi$.

Proof.

This is proved in [10].

LEMMA 8.

Let Y *and* π *be as in the hypothesis of Lemma 7. Let* (A_n) *be a sequence of families of members of* π *as in the Lemma 6. Let* (F_n) *be a sequence of infinite collections of members of* π *so that the growth* A_n^* *of the set* $A_n = \bigcup_{X \in F_n} X$ *is non-empty and disjoint with* Y *for all* $n \in N$. *Then there is a sequence* $D_1, D_2, \ldots, D_n, \ldots$ *of non-empty compact sets of* βN *so that the following hold:*

(a) $Y \cup D_n$ *is open in* βN *for all* $n \in N$.

(b) *There exists a compact open set* V_n *of* βN *so that* $D_n \subset V_n \subset (Y \cup D_n)$ *and* $V_n \cap Y$ *is a union of members of* π *for all* $n \in N$.

(c) $D_n \cap D_m = \phi$ *for all* $n, m \in N$ *and* $n \neq m$.

(d) *Given* $n, k \in N$ *and a compact open set* W *of* βN *containing* D_n *there is a member* A *of* A_k *so that* $A \subset W$.

(e) *If* $M = Y \cup (\bigcup_{n=1}^{\infty} D_n)$ *and* $\pi_o = \pi \cup \{D_n \mid n \in N\}$ *then* π_o *is a partition of* M *and* M/π_o *is a countable, locally compact,* T_2 *space.*

(f) $M \cap A_n^* \neq \phi$ *for all* $n \in N$ *where* A_n *is defined above in this Lemma.*

Proof.

First of all get compact sets $C_1, C_2, \ldots, C_n, \ldots$ as in the conclusion of Lemma 5. Put $Y_1 = Y \cup (\bigcup_{n=1}^{\infty}\ {}_n)$ and $\pi_1 = \pi \cup \{C_1, C_2, \ldots, C_n, \ldots\}$. Then (Y_1, π_1) satisfy the hypothesis of Lemma 7. If $A_n^* \cap Y_1 \neq \phi$ for all $n \in N$ then take $D_n = C_n$ for all $n \in N$. If not let n_1 be the first integer so that $A_{n_1}^* \cap Y_1 = \phi$. Apply Lemma 7 and an open set M as in the conclusion of that lemma with A_{n_1} replacing A and (Y_1, π_1) replacing (Y, π) in that lemma. Then there is a compact set F of $\beta N - N$ so that $F \neq \phi$ and $F \cap Y_1 = \phi$ and there is a compact open set W of βN so that $F \subset W \subset (F \cup Y_1)$ and $W \cap Y_1$ is a union of members of π_1 and $A_{n_1}^* \cap W \neq \phi$. A look at the proof of Lemma 5 shows that W can be further chosen so that $W \cap C_n = \phi$ for all $n \in N$. So choose an open compact subset W as above. Put $D_1 = C_1 \cup (W/Y_1)$. We define D_n in general by induction. Assume that

n is a given integer > 1 and that we have defined $D_1, \ldots, D_{n-1}$ in such a way that $A_i^* \cap (Y_1 \cup \bigcup_{i=1}^{n-1} D_i) \neq \phi$ for $i=1,2,\ldots,n-1$. Put

$Y_2 = Y_1 \cup (\bigcup_{i=1}^{n-1} D_i)$ and $\pi_2 = \pi \cup \{D_1, \ldots, D_{n-1}, C_n, C_{n+1}, \ldots\}$. If $A_i^* \cap Y_2 \neq \phi$ for all $i \in N$ then put $D_i = C_i$ if $i \geq n$. If not there is a least integer k so that $A_k^* \cap Y_2 = \phi$. Then we get a W_o as above with the condition that $W_o \cap D_i = \phi$ for $i=1,2,$ $W_o \cap C_i = \phi$ for $i \geq n$ and $W_o \cap A_k^* \neq \phi$ and $W_o \cap Y$ is a union of members of π. Put $D_n = W_o / Y_2$. Thus proceeding by induction we get $D_1, D_2, \ldots, D_n, \ldots$ as required.

LEMMA 9.

Let $\clubsuit_\infty$ be satisfied. For every countable limit ordinal α let $A_{\alpha 1}, A_{\alpha 2}, \ldots, A_{\alpha n}, \ldots$ be as in the statement of $\clubsuit_\infty$. Then, given a limit ordinal $\alpha \in [1, \Omega)$ there is a countable family F_α of subsets of $[1, \alpha)$ with the following properties:

(a) If $A \in F_\alpha$ then A is cofinal in $[1, \alpha)$.

(b) Given a limit ordinal α and $A, B \in F_\alpha$ we have $A \cap B = \phi$ unless $A = B$.

(c) If α is a limit ordinal and F_α is infinite and $F_\alpha = \{A_{\alpha 1}, A_{\alpha 2}, \ldots, A_{\alpha n}, \ldots\}$ and g_n is the least element in $A_{\alpha n}$ for $n \in N$ then we have $g_1 < g_2 < \ldots < g_n < \ldots$ and the sequence (g_n) is cofinal with α.

(d) If α is a limit ordinal and $A \in F_\alpha$ then A is discrete and closed in $[1, \alpha)$ in the usual order topology of $[1, \alpha)$.

(e) Given an uncountable subset $B \subset [1, \Omega)$ there is a countable limit ordinal α and a set $A \in F_\alpha$ so that $A \subset B$.

(f) F_α is infinite for all limit ordinals α in $[1, \Omega)$.

Proof.

The proof of this conbinatorial Lemma is long and is postponed to appear in another paper.

Hereafter, we use (CH) and $\clubsuit_\infty$ and F_α will be as in Lemma 9.

CONSTRUCTION 10. (V-PROCESS).

We put $Y_1, \pi_1, \underline{A_1}, \underline{A_{1\delta}}, B_1, B_2, \ldots, B_n, \ldots$ as in definition 4 for all $n \in N$ and $\delta \in [1, \Omega)$. Recall that given $\alpha \in [1, \Omega)$; λ_α denotes the α^{th} limit ordinal in $[1, \Omega)$. Assume that given an ordinal α in $[1, \Omega)$ such that $\alpha > 1$ we have defined $Y_\gamma, \pi_\gamma, \underline{A_\gamma}, \underline{A_{\gamma\delta}}$, for all $\gamma < \alpha$ and $\delta \in [1, \Omega)$ and B_γ for all $\gamma < \lambda_{\alpha*}$ so as to satisfy the following; where $\alpha* = \alpha$ if α is a limit ordinal and $\alpha*$ is predecessor of α otherwise:

(i) Y_γ is a dense open set of βN and π_γ is a partition of Y_γ by compact sets for all $\gamma < \alpha$.

(ii) $1 \leq \gamma < \delta < \alpha$ implies that $Y_\gamma \subset Y_\delta$ and $\pi_\gamma \subset \pi_\delta$.

(iii) If $\gamma \in [1, \alpha)$ and $A \in \underline{F}_{\lambda_\gamma}$ and $A = (\delta_1, \delta_2, \ldots, \delta_n, \ldots)$ then given τ such that $\lambda_\gamma < \tau < \lambda_\alpha$ and a compact open set W of βN so that $B_\tau \subset W$ we have that $W \supset B_{\delta_n}$ for some $n \in N$.

(iv) Y_γ/π_γ is a countable, locally compact, T_2 space for all γ in $[1, \alpha)$.

(v) $\pi_\gamma = \{B_\delta \mid \delta \in [1, \lambda_\gamma)\}$ for all $\gamma \in [1, \alpha)$.

(vi) Given $\gamma \in [1, \alpha)$ and $\delta < \lambda_\gamma$ there is a compact open set W of βN so that $B_\delta \subset W \subset Y_\gamma$ and W is a union of members of π_γ.

(vii) Given $\gamma \in [1, \alpha)$ we have that $\underline{A_\gamma}$ is the collection of all infinite families of members $C_1, C_2, \ldots, C_n, \ldots$ of π_γ so that the growth $C*$ of $C = \bigcup_{n=1}^{\infty} C_n$ is non-empty and has empty intersection with Y_γ.

(viii) Given $\gamma \in [1, \alpha)$ we have that $\underline{A_{\gamma 1}}, \underline{A_{\gamma 2}}, \ldots, \underline{A_{\gamma\delta}}, \ldots$ is a well ordering of $\underline{A_\gamma}$ by $[1, \Omega)$.

(ix) Given $\gamma \in [1, \gamma)$ and $\delta \in [1, \gamma)$ and $\beta \in [1, \omega^\gamma)$ we have that $Y_\gamma \cap A^*_{\delta\beta} \neq \phi$ where $A^*_{\delta\beta}$ is the growth of the set $A_{\delta\beta} = \bigcup_{X \in \underline{A_{\delta\beta}}} X$.

Then we define $Y_\alpha, \pi_\alpha, \underline{A_\alpha}, \underline{A_{\alpha\delta}}, B_\gamma$ as follows for $\delta \in [1, \Omega)$ and $\gamma \in [\lambda_\alpha, \lambda_{\alpha+1})$. Consider $\underline{F_{\delta\alpha}}$ and write its members as $\underline{A_1}, \underline{A_2}, \ldots, \underline{A_n}, \ldots$ Put $Y_{\alpha-} = \bigcup_{\gamma<\alpha} Y_\gamma$ and $\pi_{\alpha-} = \bigcup_{\gamma<\alpha} \pi_\gamma$. Let $\underline{C}$ denote the set of all $\underline{A_{\delta\beta}}$ so that $1 \leq \delta < \alpha$ and $1 \leq \beta < \omega^\alpha$ so that growth $A^* \cap Y_{\alpha-} = \phi$

where $A = \bigcup_{X \varepsilon \underline{A_{\delta\beta}}} X$. Now using $Y_{\alpha-}, \pi_{\alpha-}, (\underline{A_n})$, and $\underline{C}$ in Lemma 8 at the appropriate places get $D_1, D_2, \ldots, D_n, \ldots$ as in the conclusion of that lemma so that (a) - (f) of that lemma are satisfied.

Put $B_{\lambda_\alpha} = D_1$ and $B_{\lambda_\alpha + n} = D_{n+1}$ for all $n \varepsilon N$.

Put $\underline{Y_\alpha} = \bigcup_{\gamma<\alpha} Y_\gamma \cup \bigcup_{n=1}^{\infty} D_n$ and $\pi_\alpha = \{B_\gamma | \gamma \varepsilon [1, \lambda_{\alpha+1})\}$.

Put $\underline{A_\alpha}$ to be the set of all infinite families $\{B_{\gamma_1}, B_{\gamma_2}, \ldots, B_{\gamma_n}, \ldots\}$ where $\gamma_n \varepsilon [1, \lambda_{\alpha+1})$ for all $n \varepsilon N$ and such that the growth $A^* \neq \phi$ and $A^* \cap Y_\alpha = \phi$ where $A = \bigcup_{n=1}^{\infty} B_{\gamma_n}$. Let $\underline{A_{\alpha 1}}, \underline{A_{\alpha 2}}, \ldots, \underline{A_{\alpha\delta}}, \ldots$ be a well ordering of $\underline{A_\alpha}$ by $[1, \Omega)$. Finally we put $Y_\Omega = \bigcup_{\alpha<\Omega} Y_\alpha$ and $\pi_{\Omega-} = \bigcup_{\alpha<\Omega} Y_\alpha$ and $\pi_{\Omega-} = \bigcup_{\alpha<\Omega} \pi_\alpha$ and $X_o = Y_{\Omega-}/\pi_{\Omega-}$.

THEOREM 11.

X_o is a locally compact, T_2, sequentially compact, first countable, locally countable non-compact space. Further X_o is hereditarily separable.

Proof.

The proof of the properties of X_o except that of hereditary separability is exactly like that of Theorem 1.8 and 1.9 in [10]. Now we come to the hereditary separability of X_o. Let F be an uncountable subset of X_o. Then there exists a unique uncountable subset $B \subset [1, \Omega)$ so that $\phi(B_\gamma) \varepsilon F$ if and only if $\gamma \varepsilon B$ where ϕ: $Y_{\Omega-} \longrightarrow X_o$ is the natural map. Then there is an $\alpha \varepsilon [1, \Omega)$ and a member $A \varepsilon \underline{F_{\lambda_\alpha}}$ so that $A \subset B$. Let $Z_o = \{\phi(x_\gamma) | \gamma \varepsilon A\}$. Then $\overline{Z}_o/B$ is at most countable. Hence there is a countable dense subset in B. Thus the Theorem.

THEOREM 12.

Assuming (CH) + $\clubsuit_\infty$ there is a compact, T_2, C-space which is not sequential.

Proof.

The one point compactification X_o^* of X_o in Theorem 11 is easily seen to be such an example.

REMARK.

The proof of Theorem II given in the beginning of this paper is long and cannot be accommodated in this hours talk. So it will appear elsewhere.

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