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Categorial aspects are useful for topology

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CATEGORIAL ASPECTS ARE USEFUL FOR TOPOLOGY

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Praha

Under this title, a lecture by M.Hušek and the author was delivered at the Topological symposium. In the lecture, several themes were discussed. We wanted to show some examples how categorial methods and categorial point of view bring or inspire results often "purely topological".

The present paper is a part of this lecture. It consists of two themes discussed in the lecture (the other themes will appear elsewhere), namely

I. EMBEDDINGS OF CATEGORIES

and

II. HOMEOMORPHISMS OF PRODUCTS OF SPACES.

These themes concern distinct fields of problems; however, they are not independent in their methods. The first theme leads e.g. to constructions of stiff classes of spaces (see I.2) and the second one heavily uses them.

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I.

1. Let us begin with the well-known result of de Groot ([dG]) that every group is isomorphic to the group of all homeomorphisms of a topological space onto itself. In 1964, at the Colloquium on topology in Tihany, he put a problem whether any monoid (i.e. a semigroup with the unit element) is isomorphic to the monoid of all non-constant continuous mappings of a topological space into itself. Let us notice that the set of all non-constant continuous mappings does not always form a monoid, the composition of two non-constant mappings can be constant. The exact formulation is as follows. Given a monoid M, does there exist a space X such that the set of all non-constant continuous mappings of X into itself is closed under composition and this set, endowed with this composition, forms a monoid isomorphic to M ? This was solved positively in [Tr_1], the space X can even be chosen to be metrizable, or, by [Tr_5], compact and Hausdorff. The proof is based on a nice result of Z. Hedrlín and A. Pultr. They proved in [HP] that any small category (i.e. a category, the objects of which form a set) is isomorphic to a full subcategory of the category Graph of all directed graphs and all their compatible mappings. What is really presented in [Tr_1] is a construction of a functor $\mathcal{M}$ of Graph into the category Metr of all metrizable spaces and all their continuous mappings, with the following property. For any pair G, G′ of graphs,

$$f: G \longrightarrow G' \overset{\mathcal{M}}{\rightsquigarrow} \mathcal{M}(f): \mathcal{M}(G) \longrightarrow \mathcal{M}(G')$$

is a bijection of the set of all compatible mappings of G into G′ onto the set of all non-constant continuous mappings of $\mathcal{M}(G)$ into $\mathcal{M}(G')$. Since any monoid M can be considered as the set of all morphisms of a category with precisely one object, there exists, by [HP], a graph G such that M is isomorphic to the monoid of all compatible mappings of G into itself. Hence, M is isomorphic to the monoid of all non-constant continuous mappings of $\mathcal{M}(G)$ into itself. In [Tr_5], a functor $\mathcal{C}$ from the category $(\text{Graph})^{op}$, opposite to Graph, into the category Comp of all compact Hausdorff spaces is constructed such that, again, for any pair of graphs, G, G′,

$$f: G \longrightarrow G' \overset{\mathcal{C}}{\rightsquigarrow} \mathcal{C}(f): \mathcal{C}(G') \longrightarrow \mathcal{C}(G)$$

is a bijection of the set of all compatible mappings of G into G′ onto the set of all non-constant continuous mappings of $\mathcal{C}(G')$ into $\mathcal{C}(G)$. This makes it possible to obtain the analogous result for compact Hausdorff spaces.

2. These categorial methods give, as a byproduct, some results concerning stiff classes of spaces. Let us recall that a class $\mathfrak{C}$ of topological spaces is called stiff if for any X, Y $\in \mathfrak{C}$ and any continuous mapping $f: X \longrightarrow Y$ either f is constant or X = Y and f is the identity (sometimes, also the word rigid or strongly rigid is used). Let a cardinal $\mathfrak{m}$ be given, let $k(\mathfrak{m})$ be a discrete category (i.e. with no morphisms except the identities) such that its objects form a set of the cardinality $\mathfrak{m}$. Since $k(\mathfrak{m})$ is a small category, it is isomorphic to the full subcategory of Graph. Its image under $\mathcal{M}$ is a stiff set (of the cardinality $\mathfrak{m}$) of metrizable spaces. Analogously, we obtain arbitratily large stiff sets of compact Hausdorff spaces by means of the functor $\mathcal{C}$. Let us remark that L. Kučera and Z. Hedrlín proved (see [H]) that, under the following set-theoretical assumption

(M) relatively measurable cardinals are not cofinal in the class of all cardinals,

any concrete category is isomorphic to a full subcategory of Graph. A "large discrete category" is concrete, obviously. Consequently, under (M), the functor $\mathcal{M}$ (or $\mathcal{C}$) gives a stiff proper class of metrizable (or compact Hausdorff) spaces. Let us notice that a stiff proper class of paracompact spaces was constructed in [K] without any set-theoretical assumption.

3. Let us recall some usual notions about categories and functors. A functor $\Phi : K \longrightarrow H$ is called a full embedding if it is an isomorphism of K onto a full subcategory of H. Now, let H be a category of topological spaces and all their continuous mappings. Φ is called an almost full embedding if, for any pair a, b of objects of K,

$$f: a \longrightarrow b \overset{\Phi}{\rightsquigarrow} \Phi(f): \Phi(a) \longrightarrow \Phi(b)$$

is a bijection of the set of all morphisms of a to b onto the set of all non-constant continuous mappings of $\Phi(a)$ to $\Phi(b)$. A category U is called universal (or s-universal) if every concrete category (or small category, respectively) can be fully embedded in it. A category T of topological spaces and all their continuous mappings is called almost universal (or almost s-universal) if every concrete category (or small category, respectively) can be almost fully embedded in it. In this terminology, Graph is s-universal and, under (M), it is universal. Metr and Comp are almost s-universal and, under (M), they are almost universal. What V. Koubek really proved in [K] is that the category Par of all paracompact spaces is almost universal. (He starts from a result of L. Kučera and Z. Hedrlín that a rather simply defined category is universal and constructs an almost full embedding of it to the category Par.)

All the above results and their proofs and many others (for example, the investigation of topological categories with other choice of morphisms than all continuous mappings) are contained, with all the details, in the prepared monograph [PT].

4. All the above results say that there are spaces such that all non-constant continuous mappings between any pair of them have some prescribed properties. A classical question of topology is about non-constant continuous mappings into a given space. Let us recall the regular space without non-constant continuous real function of E. Hewitt [Hw] and J. Novák [N] and the following well-known generalization of H. Herrlich [Hr]. For any T_1-space Y there exists a regular space X with more than one point and such that any continuous mapping f: $X \longrightarrow Y$ is constant. Now, we can ask about the coherence of these problems. For example, let a T_1-space Y and a monoid M be given. Does there exist a space X (regular, if possible) such that any continuous mapping f: $X \longrightarrow Y$ is constant and all non-constant continuous mappings of X into itself form a monoid isomorphic to M ? A stronger assertion than the affirmative answer to this question states, is the following

Theorem. For any T_1-space Y, all regular spaces without non-constant continuous mappings into Y (and all their continuous mappings) form an almost universal category.

5. Let us sketch a general construction which gives not only the above theorem but also some further results stated in the next theorems. It is based on the combination of the method used in [Tr_1],[K] and that of [Hr],[EF],[G]. Let Φ_0 be a functor of a category K into the category Top of all topological spaces. Let, for any K-object σ, the space $\Phi_0(\sigma)$ contain a point, say σ_0, with the following property.

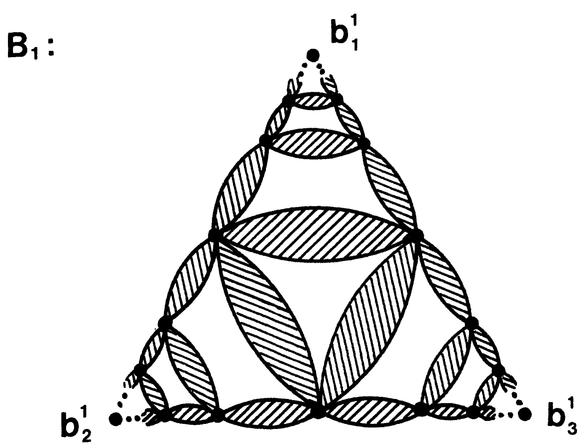
For any K-morphism m: $\sigma \longrightarrow \sigma'$, $\Phi_0(m)$ maps $\Phi_0(\sigma) \setminus \{\sigma_0\}$ into $\Phi_0(\sigma') \setminus \{\sigma'_0\}$ and σ_0 on σ'_0.

Let Q be a space with three distinguished points, say q_1, q_2, q_3. By an "iterated glueing" we obtain a new functor $\Psi : K \longrightarrow$ Top. It is defined by induction. We start with Φ_0. If Φ_n: $K \longrightarrow$ Top and σ_n in every $\Phi_n(\sigma)$ are defined, Φ_{n+1} is obtained as follows. For any

$x \in \Phi_n(\sigma) \setminus \{\sigma_n\}$, we add a copy of Q to $\Phi_n(\sigma)$ and identify q_1 with x, q_2 with σ_n; finally, we identify the q_3's of all copies of Q; the obtained point is σ_{n+1}. If m: $\sigma \longrightarrow \sigma'$ is a K-morphism, then $\Phi_n(m)$ is extended to $\Phi_{n+1}(m)$ so that the copy of Q joining x and σ_n is mapped "identically" onto the copy of Q joining $(\Phi_n(m))(x)$ and σ'_n. Ψ is the union of the functors Φ_n, $n = 0,1,2,\dots$.

6. Now, let K be a universal (or s-universal) category and let Ψ be an almost full embedding. Then the range category of Ψ is almost universal (or almost s-universal, respectively) and the spaces $\Psi(\sigma)$ have some desired properties.This is the basic idea of all the proofs. We start with the functor from the universal category into Par, constructed in [K], or from the s-universal category Graph into Metr, constructed in [Tr_1]. This is Φ_0. If Q is suitably chosen, Ψ can be proved to be an almost full embedding.
The construction of Φ_0 in [K] and [Tr_1] as well as the construction of suitable Q heavily use the existence of a Cook continuum [C], i.e. a metrizable continuum H such that, for any subcontinuum L and any continuous mapping f: $L \longrightarrow H$, either f is constant or $f(x) = x$ for all $x \in L$.

7. We sketch briefly the construction of the space Q. It depends on a given space Z (when Z is a regular totally disconnected space such that any continuous mapping of Z into a given T_1-space Y does not distinguish two points q_1, q_2, we obtain the previous theorem, but other choices of Z are used, too). Let $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$ be countable sets of non-degenerate subcontinua of a Cook continuum H such that $\mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2$ is pairwise disjoint. $\mathcal{A}$ is used for the construction of Φ_0 as in [K] or [Tr_1]. Then we construct two spaces B_1 and B_2 like in the following figure:



where the 's are distinct members of $\mathcal{B}_1$ (in any of them, two distinct points are chosen for the merging). B_2 and b_1^2, b_2^2, b_3^2 are constructed analogously, by means of $\mathcal{B}_2$. We construct Q starting from the space $Z \vee Z'$, where Z is the given space, Z' is a discrete space of the same cardinality, $\vee$ denotes a disjoint union as closed-and-open subsets and $z \longrightarrow z'$ is a bijection of Z onto Z'. For any $z \in Z$ we add a copy of B_1, where we identify b_1^1 with z and b_2^1 with z'. Let $R \subset \subset Z' \times Z'$ be the binary relation described in [VHP]. For any $(y_1, y_2) \in \in R$, we join y_1 with y_2 by a copy of B_2, i.e. we identify y_1 with b_1^2 and y_2 with b_2^2; finally, we identify all the points b_3^1 and b_3^2 for all the copies of B_1 and B_2. The point obtained by this last identification is q_3, q_1 and q_2 are two distinct points of Z.

8. The theorems stated in 9. and 10. are obtained by this construction if we choose a suitable Z. (The particular choice of Z is always given after the theorem.) In all these cases, it can be seen easily that the spaces $\Psi(\sigma)$ have the required properties. On the other hand, the proof that Ψ is really an almost full embedding, which is the heart of the matter, is more complicated and rather technical.

9. Let V be a topological space. We say that <u>a space X contains V many times</u> if for any $v \in V$ and any $x \in X$ there exists a homeomorphism h of V onto a closed subspace of X such that $h(v) = x$. In the following theorems, speaking about categories of topological spaces, we always mean these spaces and all their continuous mappings. <u>All spaces are supposed to be T_1-spaces.</u>

<u>Theorem.</u> Let V be a paracompact (or normal or completely regular) totally disconnected space. Then all paracompact (or normal or completely regular) spaces, containing V many times, form an almost universal category.

<u>Theorem.</u> Let V be a metrizable totally disconnected space. Then all metrizable spaces, containing V many times, form an almost s-universal category and, under (M), they form an almost universal category.

For the proof of these theorems, we use Z in the above construction as follows. We take a copy of V, say V(v), for any point $v \in V$, and identify all these points v, each in its copy V(v). The obtained point is q_1, $q_2 \in Z \setminus \{q_1\}$ is arbitrary. For the second theorem, all the identifications in the definition of the functor Ψ must be done "metrically".

10. For separation properties weaker than the complete regularity, the construction gives a much stronger result. We can omit the assumption that the given space V is totally disconnected and, simultaneously, continuous mappings in a given space can still be required to be constant. More precisely, the following theorem holds.

Theorem. Let a space Y be given. Let V be a space (or Hausdorff or regular). Then all the spaces (or Hausdorff spaces or regular spaces, respectively) X containing V many times and such that any continuous mapping $f: X \longrightarrow Y$ is constant, form an almost universal category.

For the proof of this theorem, we use Z in the above construction as follows. We take a copy V(v) of V, for any $v \in V$, and identify these points v, as in 9. Denote the obtained space by W, its point obtained by the identifications of the v's by w. Now, let U be a totally disconnected regular space and q_1, q_2 two its distinct points such that, for any continuous mapping f of U into any T_1-space of the cardinality smaller than exp $(\aleph_0 \cdot \text{card } Y \cdot \text{card } W)$, $f(q_1) = f(q_2)$. (The space constructed in [Hr] or [G] has really only one-point components, q_1 and q_2 are, of course, in one quasicomponent.) Z is a space obtained from a disjoint union of W and U by the identification of w and q_1.

11. Let us show some "purely topological" immediate consequences of the above theorems. By the last theorem,

there exists a stiff proper class of regular spaces, in which any point lies on an arc.

Another application: since for any set X there exists $R \subset X \times X$ such that the graph (X,R) has no non-identical endomorphism (see [VPH]),

any totally disconnected space V can be embedded as a closed subspace in a space X without non-constant non-identical continuous mappings into itself such that card $X = 2^{\aleph_0} \cdot$ card V and X is completely regular or normal or paracompact or metrizable whenever V has this property.

Analogously,

for any space V and any cardinal $\alpha >$ card V there exists a space X without non-constant non-identical continuous mappings into itself such that card $X = 2^{\aleph_0} \cdot \alpha$, X contains V as a closed subspace and X is Hausdorff or regular whenever V has this property.

Spaces without non-constant non-identical continuous mappings into itself are considered in [KR], where for any infinite cardinal α such Hausdorff space X with card X = α is constructed.

12. The described construction does not "work" for compact spaces. Nevertheless, the following theorem holds.

Theorem. Let V be a totally disconnected compact Hausdorff space. Then all connected compact Hausdorff spaces, containing V many times, form an almost s-universal category. Under (M), they form an almost universal category.

Here, the proof starts from a modification of the almost full embedding of $(\mathrm{Graph})^{op}$ into Comp, described in [Tr_5], and the "iterated glueing" must be done in a different way. The full proof will appear in [Tr_6], where also the proofs of the previous embedding theorems will be given in more detail.

II.

1. In 1957, W. Hanf [H] constructed a Boolean algebra B isomorphic to $B \times B \times B$ but not to $B \times B$. The analogous result for Abelian groups was proved by A.L. Corner in 1963 (see [Cr]). The analogous problem can be investigated in an arbitrary category. Let $\mathbb{K}$ be a category with finite products. Given a natural number $n \geq 3$, denote by

$$\mathbb{K}(n)$$

the class of all objects X of $\mathbb{K}$ such that

X is isomorphic to $X \times \ldots \times X$ (n-times) and $X \times \ldots \times X$ (k-times) is not isomorphic to $X \times \ldots \times X$ (k′-times) whenever $1 \leq k < k' \leq n - 1$.

Let us consider $\mathbb{K}$ to be the category of topological spaces. By [Tr_2], for every n, $\mathbb{K}(n)$ contains a locally compact separable metrizable space. A large part of the method of the proof is categorial, it admits not only a categorial formulation but also an application to other familiar categories. This is done in [Tr_3], where the analogous result is shown also for uniform and proximity spaces, graphs, small categories and some types of partial algebras and unary algebras.

2. Now, we strengthen the above result as follows.

__Theorem.__ Let $\mathbb{K}$ be the category of topological spaces. Let $\mathbb{C}$ be a class of spaces such that

(a) $\mathbb{C}$ contains all metrizable continua;

(b) $\mathbb{C}$ is closed under finite products and countable coproducts (= disjoint unions as clo-open subsets).

Then for any $n \geq 3$ and any X in $\mathbb{C}$ there exist $2^{\aleph_0}$ non-homeomorphic spaces in $\mathbb{C} \cap \mathbb{K}(n)$ such that each of them contains X as a closed subspace and its cardinality is equal to $2^{\aleph_0} \cdot \operatorname{card} X$.

__Proof.__ a) If Y is a space, denote by Y^0 a one-point space, $Y^1 = Y$, $Y^{n+1} = Y \times Y^n$. Denote by N the set of all non-negative integers and by N^N the set of all functions on N with values in N. Let $\{K(x) \mid x \in N \cup \{\infty\}\}$ be a countable stiff set (see I.2) of metrizable continua. For any $\ell \in N^N$ put $K_\ell = \prod_{x \in N} (K(x))^{\ell(x)}$. By [$Tr_3$],

(*) K_ℓ is not homeomorphic to $K_{\ell'}$ whenever $\ell \neq \ell'$.

b) For $\ell, \ell' \in N^N$ define $\ell + \ell'$ by $(\ell + \ell')(x) = \ell(x) + \ell'(x)$. For $A, B \subset N^N$ define $A + B = \{a + b \mid a \in A, b \in B\}$. If $n = 1,2,\ldots$, put

$n \cdot A = A + \ldots + A$(n-times). Let $n \geq 3$ be given. By [Tr_3], there exists a countable set $A \subset N^N$ such that

(i) for any $a \in A$, $a(x) \neq 0$ for infinitely many $x \in N$;

(ii) $A = n \cdot A$;

(iii) if $1 \leq k < k' \leq n - 1$, then $k \cdot A \cap k' \cdot A = \emptyset$.

c) Let a space X in $\mathbb{C}$ be given. Put $Z = X \times K(\infty)$, hence $Z \in \mathbb{C}$. Put

$$\widetilde{Y} = \coprod_{\substack{x \in N \\ a \in A}} (Z^x \times K_a), \qquad \widetilde{Y}_0 = \coprod_{a \in A} (Z^0 \times K_a),$$

where $\coprod$ denotes coproduct. Let Y (or Y_0) be a coproduct of $\aleph_0$ copies of $\widetilde{Y}$ (or $\widetilde{Y}_0$, respectively). Clearly, Y contains X as a closed subspace and card $Y = 2^{\aleph_0} \cdot$ card X. Since Y contains $\aleph_0$ copies of any $Z^x \times K_a$, Y is homeomorphic to Y^n, by (ii).

d) Let us notice that any continuous mapping of $K(\infty)$ into K_ℓ is constant for any $\ell \in N^N$. Hence Y_0 consists precisely of all components C of Y such that any continuous mapping of $K(\infty)$ into C is constant. Consequently, Y_0^k is homeomorphic to $Y_0^{k'}$ whenever Y^k is homeomorphic to $Y^{k'}$, $1 \leq k \leq k' \leq n - 1$. By (iii), this is possible only when $k = k'$. Thus, the space Y has all the required properties.

e) Now, we show that there are many such spaces. Let $\mathfrak{S}$ be a system of infinite subsets of N such that card $\mathfrak{S} = 2^{\aleph_0}$ and, for any distinct $S_1, S_2 \in \mathfrak{S}$, $S_1 \cap S_2$ is finite. Let $\psi_S : N \longrightarrow S$ be a bijection. Construct Y(S) by means of the spaces $\{K(\psi_S(x)) \mid x \in N\}$ quite analogously as Y by means of $\{K(x) \mid x \in N\}$. By (i) and (*), $Y(S_1)$ is not homeomorphic to $Y(S_2)$ whenever S_1 and S_2 are distinct elements of $\mathfrak{S}$.

3. The conditions (a),(b) are not too restrictive, the theorem can be applied e.g. for the class of all spaces, all T_1-spaces, Hausdorff, regular, completely regular, metrizable, σ-compact, realcompact (or E-compact whenever E contains an infinite closed discrete subset and an arc), locally metrizable, spaces with the first or second axiom of countability, separable (or with a density character equal to a given cardinality) and many others. On the other hand, the important class of compact Hausdorff spaces does not satisfy them. Nevertheless, the following theorem holds.

Theorem. For any $n \geq 3$ and any compact Hausdorff (or compact metrizable) space X, there exists a compact Hausdorff (or compact metrizable) space in $\mathbb{K}(n)$ which contains X as a closed subspace.

Outline of the proof. We may suppose that the given space is a cube (or a Hilbert cube). Construct Y analogously as in the previous proof. Let T be a compactification of Y. Since Y is a coproduct of compact connected spaces, $y \in T$ is in Y iff y has a connected neighbourhood in T. Consequently, Y^k is homeomophic to $Y^{k'}$ whenever T^k is homeomorphic to $T^{k'}$ ($1 \leq k \leq k' \leq n-1$). This is possible only when $k = k'$. Thus, it is sufficient to construct a compactification T of Y such that T is homeomorphic to T^n. The construction will be given in two steps.

a) First, we choose a homeomorphism h of Y onto Y^n and find a compactification T_0 of Y such that the following diagram commutes:

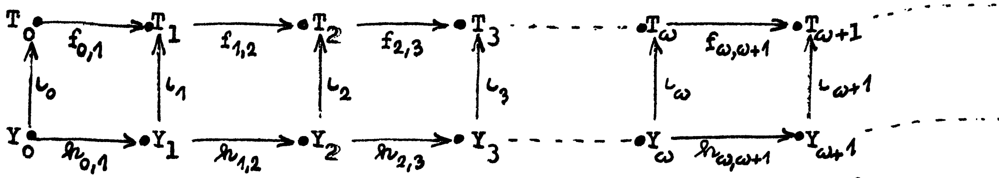
$$\begin{array}{ccc} T_0 & \xrightarrow{f} & T_0^n \\ \uparrow{\scriptstyle \iota_0} & & \uparrow{\scriptstyle \iota_0^n} \\ Y & \xrightarrow[h]{} & Y^n \end{array} ,$$

where ι_0 is the embedding and f is a continuous mapping (if $g: P \longrightarrow Q$ is a mapping, we denote by $g^n: P^n \longrightarrow Q^n$ the mapping defined by $g^n(p_1,\dots,p_n) = (g(p_1),\dots,g(p_n))$). This is easy for compact Hausdorff spaces; we put $T_0 = \beta Y$ and f is a continuous extension of $\iota_0^n \circ h$. Now, we construct a metrizable compactification T_0 for metrizable Y. Let H be the Hilbert cube, $\varkappa : Y \longrightarrow H$ an embedding. Define $\lambda : Y \longrightarrow \prod_{k=0}^{\infty} H^{n^k}$ by

$$\lambda(y) = (\varkappa(y), \varkappa^n(h(y)), \varkappa^{n^2}(h^n(y)), \dots, \varkappa^{n^{k+1}}(h^{n^k}(y)), \dots)$$

and $\tau : \prod_{k=0}^{\infty} H^{n^k} \longrightarrow \prod_{k=1}^{\infty} H^{n^k}$ by $\tau(z_0, z_1, z_2, \dots) = (z_1, z_2, \dots)$, where $z_i \in H^{n^i}$. Denote $R = \{1,2,\dots,n\}$. Then any $z \in (\prod_{k=0}^{\infty} H^{n^k})^n$ can be expressed as $z = (((z_{i,j,k} \mid j \in R^k) \mid k = 0,1,2,\dots) \mid i \in R)$. Define a homeomorphism $\sigma : (\prod_{k=0}^{\infty} H^{n^k})^n \longrightarrow \prod_{k=1}^{\infty} H^{n^k}$ by $\sigma(z) = ((z_{i,j,k} \mid (i,j) \in R \times R^k) \mid k = 0,1,2,\dots)$. One can verify that $(\sigma^{-1} \circ \tau) \circ \lambda = \lambda^n \circ h$. Then define T_0 as the closure of $\lambda(Y)$ in $\prod_{k=0}^{\infty} H^{n^k}$ and f as the corresponding domain-range-restriction of $\sigma^{-1} \circ \tau$.

b) Now, we consider the following diagram.



where $Y_0 = Y$, $h_{0,1} = h$, $f_{0,1} = f$ are as in a) and $T_{i+1} = T_i^n$, $\iota_{i+1} = \iota_i^n$, $h_{i+1,i+2} = (h_{i,i+1})^n$, $f_{i+1,i+2} = (f_{i,i+1})^n$, Y_ω and T_ω are colimits in the category of all Hausdorff spaces of the preceding chains (hence, T_ω is a compact Hausdorff space; it is metrizable whenever T_0 is metrizable). The proof that ι_ω is a homeomorphism of Y_ω into T_ω is omitted as well as the definition of $h_{\omega,\omega+1}$ and $f_{\omega,\omega+1}$ whenever ω is a limit ordinal (this definition is "natural", use the fact that $Y_{i+1} = Y_i^n$, $T_{i+1} = T_i^n$). All $h_{\alpha,\beta}$ are homeomorphisms of Y_α onto Y_β, all $f_{\alpha,\beta}$ are surjective continuous mappings. Since all the T_α's are quotients of T_0, this process must stop, i.e. $f_{\alpha,\alpha+1}$ must be a homeomorphism for some ordinal α. Then T_α is a compactification with the required properties.

4. The proofs of all the above theorems are based on the stiff set $\{K(x) \mid x \in N\}$ of non-degenerate continua. Thus, none of the constructed spaces is zero-dimensional. Nevertheless, the following theorem holds (the proof will appear in [TK]).

Theorem. For any $n \geq 3$, any Boolean space can be embedded into a Boolean space from $\mathbb{K}(n)$.

5. Let us sketch a more general setting of the above field of problems. Let $\mathbb{K}$ be a category with finite products, let $(S,+)$ be a commutative semigroup. Any mapping

$$r: S \longrightarrow \mathrm{obj}\,\mathbb{K}$$

is called a representation of the semigroup by products in $\mathbb{K}$ provided that for any $s_1, s_2 \in S$, $r(s_1 + s_2)$ is isomorphic to $r(s_1) \times r(s_2)$ and $r(s_1)$ is not isomorphic to $r(s_2)$ whenever $s_1 \neq s_2$.
Hence, any object X from $\mathbb{K}(n)$ generates a representation of the finite cyclic group of the order $n-1$. In [Tr_3], a general method is described for the representation of any semigroup $\exp N^M$ (here, N^M is the semigroup of all functions on M with values in N, $\exp N^M$ is the semigroup of all its subsets) in several familiar categories, including the category of topological or uniform or proximity spaces. By [Tr_4], any commutative semigroup S can be embedded into $\exp N^{\aleph_0 \cdot \mathrm{card}\, S}$.

Hence, any commutative semigroup S has a representation by products of topological spaces. These spaces can be chosen to be coproducts of continua, by [Tr_5], or coproducts of Boolean spaces, by [AK]. The results presented in I. of this paper imply the following assertions as an immediate consequence.

Given a T_1-space X (or Hausdorff or regular), any commutative semigroup has a representation by products of T_1-spaces (or Hausdorff or regular) containing X many times.

Given a totally disconnected Tichonov space X, any commutative semigroup has a representation by products of Tichonov spaces containing X many times.

The method, used in II.2, can be used to prove easily the following assertion.

Let $\mathfrak{C}$ be a class of spaces containing all continua and closed under finite products and arbitrary coproducts. Let X be a space, let $\mathfrak{C}(X)$ be the class of all spaces from $\mathfrak{C}$, which contain X as a closed subspace. If $\mathfrak{C}(X) \neq \emptyset$, then any commutative semigroup has a representation by products of spaces from the $\mathfrak{C}(X)$.

In [AK], the semigroups $(\exp N^N)^M$ are represented by products such that only countable products of spaces of the basic system $\{K(x) \mid x \in N \times M\}$ are used. This makes it possible to represent the class of all semigroups, embeddable in $(\exp N^N)^M$ for some set M, by products of metrizable spaces. This class of semigroups contains all Abelian groups. Thus, the method of [AK] and the results presented here can be used to prove easily the following assertions.

Given a totally disconnected metrizable space X, any Abelian group has a representation by products of metrizable spaces containing X many times.

Let $\mathfrak{C}$ be a class of spaces containing all complete metric semicontinua and closed under finite products and arbitrary coproducts. Let X be a space, $\mathfrak{C}(X)$ the class of all spaces from $\mathfrak{C}$ which contain X as a closed subspace. If $\mathfrak{C}(X) \neq \emptyset$, then any Abelian group has a representation by products of spaces from $\mathfrak{C}(X)$.

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