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HOMEOMORPHISMS OF 2-DIMENSIONAL CONTINUA

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The purposes of these investigations are

(1) to establish apparatus for exhibiting homeomorphisms between spaces and of a space onto itself where isotopy-type mappings are not possible because the spaces are not locally homologically trivial,

(2) to establish apparatus for characterizing compact metric spaces in terms of sequences of coverings particularly where the local structure of the space has not previously been identified, and

(3) to characterize various classes of homogeneous locally connected metric continua.

Specific results are obtained for some 2-dimensional continua. Further results concerning other 2-dimensional and some higher-dimensional continua can be anticipated.

The particular method of description to be used is suggested by a refinement sequence of finite closed non-overlapping connected partitionings of the space, one in which the elements are homeomorphic to each other and admit general sequential type descriptions.

A triple of sequences $(\{F_i\}, \{\varphi_i\}, \{\alpha_i\})$ is an inverse incidence system provided that, for each i,

(1) F_i is a finite set,

(2) φ_i is a map of F_{i+1} onto F_i,

(3) α_i is a reflexive and symmetric binary (incidence) relation on F_i, and

(4) if $(a, b), (b, c) \in \alpha_{i+1}$ then $(\varphi_i(a), \varphi_i(c)) \in \alpha_i$.

The pair $(\{F_i\}, \{\varphi_i\})$ is an inverse system whose inverse limit L is a zero-dimensional compact metric space. In the uses we make, the sets $\varphi_i^{-1}(f)$ will be non-degenerate and thus L will be a Cantor set.

Let R be a binary relation on L defined by $(\{f_i\}, \{f_i^0\}) \in R$ provided that, for each i, $(f_i, f_i^0) \in \alpha_i$. Using condition (4) of the definition above, it follows that R is an equivalence relation and that the set L of equivalence classes defined by R is an upper semi-continuous decomposition $\tilde{L}$ of L. The collection $\tilde{L}$ (topologized) is called the inverse

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incidence limit of $(\{F_i\}, \{\varphi_i\}, \{\alpha_i\})$. It is interesting to note that only binary incidence relations are needed for this structure.

The basic problem is to determine conditions on two sequences under which their inverse incidence limits are homeomorphic.

It is easy to see that the inverse incidence limit of an induced inverse incidence system obtained by taking subsequences of the original sequences is canonically homeomorphic to the original inverse limit.

It is routine but lengthy to define what might be called an amalgamation-refinement of an inverse incidence system which will itself be an inverse incidence system whose inverse incidence limit is canonically homeomorphic to that of the original. The basic argument is to start with two inverse incidence systems (subject to many extra conditions) and to construct inductively amalgamation-refinements of these which admit identifications with each other so that the two new (and hence the two original) inverse incidence limits are homeomorphic to each other.

This apparatus does not directly lead to homogeneity results because singular points (those in non-degenerate equivalence classes of the definition of inverse incidence limit) cannot be made by amalgamation-refinement procedures to correspond to non-singular points.

To handle homogeneity (and more general questions) we note that the mapping from L to $\tilde{L}$ can be factored through various upper semi-continuous decompositions of L. A necessary and sufficient condition that such a factoring W induce an (equivalent) inverse incidence system is that W be zero-dimensional.

The 2-dimensional cases. A finite collection G of simple closed curves is called a *κ-collection* if

(1) the intersection of any two is an arc or is null,
(2) the intersection of any three is a point or is null,
(3) G^* (the union of the elements of G) is connected, and
(4) except for a finite point set, each point of G^* is in exactly two elements of G.

A subcollection G^1 of a κ-collection G is called a *λ-collection* if $G \setminus G^1$ is a non-null collection of disjoint elements of G. In such case the union of the elements of $G \setminus G^1$ is denoted $B(G^1)$ and is called the boundary of G^1.

If G^1 is a λ-collection then $G^{1*} = G^*$ and hence G^{1*} is connected. If G^1 is a λ-collection then there is a unique κ-collection containing it.

A λ-collection whose boundary is a single simple closed curve is called a *μ-collection.*

Let G^1 be a μ-collection. If for any two arcs t_1 and t_2 such that (1) $t_1 \cup t_2 = B(G')$ and (2) $t_1 \cap t_2$ is a set of two points each in two elements of G^1, there exist two disjoint μ-collections X_1 and X_2 such that $X_1 \cup X_2 = G^1$, $X_1^* \supset t_1$, and $X_2^* \supset t_2$, then G^1 is said to be a *ν-collection.*

We are interested in two propositions which may or may not hold in the κ-extension G of a given μ-collection.

I. There exists an arc $\alpha \subset G^*$ such that (a) α is not in any element of G, (b) for some $g \in G$, $\alpha \cap g$ is the set of endpoints of α, and (c) α does not separate G^*.

II. The elements of G may be assigned orientations so that each arc which is the intersection of two elements of G inherits opposite orientations from these two simple closed curves.

A μ-collection whose κ-extension satisfies I and II is called a T-collection (toroidal collection).

A μ-collection whose κ-extension satisfies I but does not satisfy II is called a P-collection (projective collection).

Let $(\{F_i\}, \{\varphi_i\}, \{\alpha_i\})$ be an inverse incidence system where, for each i,

1. F_i is a k-collection,
2. for any $f \in F_i$, $\varphi_i^{-1}(f)$ is a ν-collection whose boundary is canonically identified with f, and
3. if $a, b \in F_i$, then $(a, b) \in \alpha_i$ if and only if $a \cap b \neq 0$.

The inverse incidence limit (I. I. L.) of such a sequence can be identified as follows:

1. If for each i and $f \in F_i$, $\varphi_i^{-1}(f)$ is a T-collection then the I. I. L. is called an orientable T-sphere or a non-orientable T-sphere according as F_1 satisfies or does not satisfy proposition II.
2. If for each i, and $f \in F_i$, $\varphi_i^{-1}(f)$ is a P-collection, then the I. I. L. is called a P-sphere.

The basic theorems are

Theorem 1. *Every two P-spheres are homeomorphic to each other. A P-sphere is homogeneous and 2-dimensional.*

Theorem 2. *Every two orientable T-spheres are homeomorphic to each other. An orientable T-sphere is homogeneous and 2-dimensional.*

Theorem 3. *Every two non-orientable T-spheres are homeomorphic to each other. A non-orientable T-sphere is homogeneous and 2-dimensional.*

These results together with known results for 2-manifolds almost classify continua with the properties that they are homogeneous and have bases for which every element has a simple closed curve boundary which separates (and separates locally) into two connected pieces. The one-dimensional universal curve also has these properties. The assumption of the sequential structure is, therefore, needed.

The k, λ, μ, and ν-collection definitions can be abstracted. In particular they can be changed so that the elements of F_i are universal curves with the resulting set being homogeneous and characterized.

It seems likely that similar results can be obtained for boundaries which are universal plane curves if the collections are orientable.

The basic pattern of argument is the same for Theorems 1, 2 and 3 as well as for the further propositions just suggested. Abstractions (now being sought) of the conditions and of the argument may produce a somewhat general theorem leading to a "machine" for establishing characterizations and homogeneity theorems for rather broad classes of applicable spaces.

We assume two inverse incidence systems $(\{F_i\}, \{\varphi_i\}, \{\alpha_i\})$ and $(\{G_i\}, \{\nu_i\}, \{\beta_i\})$ with similar structures defining them (such as tho seof Theorems 1, 2 or 3). We consider the creation of compatible amalgamation-refinement sequence systems. We may assert that the elements of F_1 may be so ordered $f_1, \ldots, f_k$ that for any $j < k$, the set $f_j, \ldots, f_k$ is a λ-collection. Then we proceed to match the elements $(f_1, \ldots, f_k)$ of F_1 with amalgams of elements of some G_n (or more properly the boundaries of such amalgams) so that incidence properties are preserved under the matching process. To do the matching we proceed inductively.

The pairs f_1 and $\{f_2, \ldots, f_k\}$ are such that $\{f_2, \ldots, f_k\}$ is a λ-collection. We choose any element g'_1 of G_1. The set of other elements of G_1 is a λ-collection. Further, g'_1 intersects the boundary of this collection in the same way that f_1 intersects $B(\{f_2, \ldots, f_k\})$. Proceeding by induction we then consider f_j and $\{f_{j+1}, \ldots, f_k\}$ and we are able to construct g'_j in some G_m such that g'_i intersects the elements of $\{g'_1, \ldots, g'_{j-1}\}$ in the same way as f_j intersects the elements of $\{f_1, \ldots, f_{j-1}\}$ respectively whereas the "complement" of $g'_1, \ldots, g'_j$ is a λ-collection whose boundary intersects $g'_1, \ldots, g'_j$ in the same way as $B(\{f_{j+1}, \ldots, f_k\})$ intersects $f_1, \ldots, f_j$.

Having established $(g'_1, \ldots, g'_k)$ as amalgams of some G_n, we may then proceed to consider the sets of G_{n+1} which project onto the simple closed curves of the set of which g'_1 is the boundary. We then order these and repeat the previous process in reverse remembering the intersections with $g'_2, \ldots, g'_k$ as we go along. Having finished the procedure for g'_1 we then consider g'_2. It is clear that the process admits iteration (modulo lemmas asserting that the individual steps can be taken). In this way, playing back and forth, we can proceed to set up our two compatible amalgamation-refinement sequence systems. It is worth noting that the kinds of conditions we use (and hope for) here must involve reduction processes on each of the two sequence systems. In this way our argument differs from some of those concerning Euclidean-type spaces where special properties of Euclidean space might enable us to use only a one-sided argument.