UNIFORM DIMENSION OF MAPPINGS

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By the dimension dim of a mapping $f : X \to Y$, where X, Y are topological spaces, the number $\sup \{\dim f^{-1}[y] \mid y \in Y\}$ is usually understood (and similarly with ind instead of dim). Some authors considered in a certain sense stronger definitions of the dimension of mappings for metric spaces, e.g. uniformly zero-dimensional mappings [2] or, as a generalization, the strong dimension of mappings [4]. We define the uniform dimension of uniformly continuous mappings for uniform spaces. It is closely connected with the uniform dimension Δd (see [1]). Further, all mappings are supposed to be uniformly continuous, uniformities are considered as systems of entourages of the diagonal.

Definition. Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be uniform spaces, $f : X \to Y$ a mapping. The *uniform dimension* of f, denoted by $\Delta d f$, is defined as the smallest non-negative integer n with the following property: for each U in $\mathcal{U}$ there exist V in $\mathcal{V}$ and W in $\mathcal{U}$ such that, if M is a subset of Y and $M \times M \subset V$, then there exists a collection $\mathcal{K}$ of subsets of X such that $\mathcal{K}$ is a W-cover of $f^{-1}[M]$, $K \times K \subset U$ for each K in $\mathcal{K}$, and each point x of $f^{-1}[M]$ is contained in at most $n + 1$ sets of $\mathcal{K}$. If such a number does not exist we set $\Delta d f = \infty$.

If f is a mapping of a non-void uniform space X into a one-point space then $\Delta d f$ is equal to the mentioned Δd-dimension of the space X (and therefore we use the same symbol Δd).

If g is a restriction of a mapping f then $\Delta d\, g \leqq \Delta d f$, if g is the restriction of f to a dense subspace then $\Delta d\, g = \Delta d f$. If p is the canonical projection of a non-void product $X \times Y$ onto X then $\Delta d\, p = \Delta d\, Y$.

The main results may be stated as follows.

Theorem 1. *Let X, Y, Z be uniform spaces, $f : X \to Y$, $g : Y \to Z$. Then* $\Delta d(g \circ f) \leqq \leqq \Delta d f + \Delta d\, g$.

Theorem 2. *Let X, Y be uniform spaces, $f : X \to Y$. Then* $\Delta d\, X \leqq \Delta d\, Y + \Delta d f$.

Theorem 3. *Let $\{X_\alpha \mid \alpha \in A\}$, $\{Y_\alpha \mid \alpha \in A\}$ be families of uniform spaces and $bf_\alpha \mid \alpha \in A\}$ a family of mappings, $f_\alpha : X_\alpha \to Y_\alpha$. Let $f : \prod\{X_\alpha \mid \alpha \in A\} \to \prod\{Y_\alpha \mid \alpha \in A\}$ $\{$e defined by the formula $f\{x_\alpha\} = \{f_\alpha x_\alpha\}$. Then* $\Delta d f \leqq \sum \Delta d f_\alpha$.

If X is a uniform space and (R, ϱ) is a metric space, we shall denote by $C_u(X, R)$ the set of all uniformly continuous mappings of X into R, endowed with the distance σ defined by

$$\sigma(f, g) = \min\left(1, \sup\{\varrho(fx, gx) \mid x \in X\}\right).$$

If R is complete then $C_u(X, R)$ is also a complete metric space. The following theorem characterizes the dimension Δd of pseudometric spaces by means of mappings into Euclidean spaces.

Theorem 4. *Let P be a pseudometric space, k, n integers, $0 \leqq k \leqq n$. Then the following properties are equivalent:*

(1) $\Delta d\, P \leqq n$,

(2) *there exists a mapping $f : P \to E^{n-k}$ with $\Delta d f \leqq k$,*

(3) *the set of all mappings $f : P \to E^{n-k}$ with $\Delta d f \leqq k$ is a dense G_δ-set in the space $C_u(P, E^{n-k})$.*

The assumption of pseudometrizability of P is essential. Thus every metric space with finite dimension Δd can be mapped by a uniformly zero-dimensional mapping into a compact space.

Nevertheless this assertion does not hold for arbitrary metric spaces. Indeed, suppose that every metric space admits of such a mapping. Then, according to Theorem 3, this is true for every uniform space. On the other hand, it can be proved that the δd-dimension (see [3] or [1]) of a space admitting of such a mapping is equal to its Δd-dimension. This is a contradiction since these dimensions need not coincide for an arbitrary uniform space.

Theorems 2 and 4 are analogous to well-known Hurewicz theorems. We also obtain some results for the dimension dim as we have the following

Theorem 5. Let X, Y be compact Hausdorff spaces, $f : X \to Y$. Then $\dim f = \Delta d f$.

A paper containing the proofs of all theorems is intended for publication in Matematičeskiĭ Sbornik.

References

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