Milan Sekanina

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TOPOLOGIES COMPATIBLE WITH ORDERING

M. SEKANINA

Brno

The partially ordered sets have many times been an object of research of topologists. The interval topology on certain linearly ordered sets yields important examples of topological spaces. Many topologies for partially ordered sets have been defined – especially for lattices – (Birkhoff, Frink, Naito, Rennie). On the contrary, the concept of continuity of a relation has been far less developed than the concept of the continuity of algebraic operations. As far as I know, a general concept of the compatibility of a topology with an order has been dealt with, mainly in the papers [1], [2], [3], [6]. In [4] we dealt with two concepts of the compatibility of a topology and a partial order which are generalizations of the concept of Eilenberg. Now, I shall deal with one of them.

Definition 1. *Let A be a partially ordered set and u a topology on A. We say that u is compatible with the ordering of A, if u is a T_1-topology and if for every pair $a, b, a \in A, b \in A, a < b$, there exist a neighbourhood O_1 of a point a and a neighbourhood O_2 of a point b so that*

$$x \in O_1 \Rightarrow x < b \quad \text{or} \quad x \parallel b,$$
$$y \in O_2 \Rightarrow y > a \quad \text{or} \quad y \parallel a$$

hold.

In the sequel $K(A)$ means the set of all topologies on A satisfying Kuratowski's axioms. If (A, π) is a partially ordered set (partial ordering is denoted by π), $S(A, \pi)$ means the set of all topologies u on A compatible with π.

First of all there hold the following theorems (for proofs see [4]).

Theorem 1. *Order-, interval- and ideal- (see [5]) topology are compatible with the given ordering.*

Theorem 2. *Let (A, u) be a topological lattice, u be a T_1-topology. Then u is compatible with the lattice order of the set A.*

From these theorems one can see that the most important cases of topologies on partially ordered sets are covered by the concept of compatibility.

Now we shall turn to the study of $S(A, \pi)$.

Theorem 3. *$u \in S(A, \pi)$, v finer than $u \Rightarrow v \in S(A, \pi)$.*

Theorem 4. *A discrete topology on A is compatible with every order on A.*

Proofs are evident.

Definition 2. *Let (A, π) be a partially ordered set. We say that (A, π) is topologically discrete if $S(A, \pi)$ consists only of the discrete topology. We say that (A, π) is topologically trivial if $S(A, \pi) = K(A)$.*

Theorem 5. *A set (A, π) is topologically trivial exactly when for $x < y$ $Z_1 = \{z : z \geqq y\}$, $Z_2 = \{z : z \leqq x\}$ are finite sets.*

Proof. The coarsest topology w of $K(A)$ has as open sets $\emptyset$ and the sets X, for which $A - X$ is finite. Let for a certain pair x, y Z_1, for example, be infinite. Then, every neighbourhood O of the point x intersects Z_1. So $w \notin S(A, \pi)$.

The contrary is evident.

Corollary of the Theorem 5. *Let (A, π) be an infinite distributive lattice. Then (A, π) is not topologically trivial.*

Proof. If A has an infinite chain as its subset, the assertion is clear.

Now, let us admit that only finite chains are subsets of A. Then (A, π) possesses the greatest element 1 and the least element 0. As the Jordan-Dedekind condition for A is valid, all maximal chains among 0, 1 have n elements for a certain n. It is quite clear that $n \leqq 3$ implies A is finite. Let us admit that $n < N$ implies A finite. Let $0, a_2, \ldots, a_{N-1}, 1$ be one of the maximal chains. Then the interval $[0, a_{N-1}]$ is finite and $A - [0, a_{N-1}]$ is infinite. There exist two different elements b, c such that $b, c \in A - [0, a_{N-1}]$, $b \vee a_{N-1} = c \vee a_{N-1} = 1$ and $b \wedge a_{N-1} = c \wedge a_{N-1}$, which contradicts the distributivity of A.

The following theorem may be of some interest.

Theorem 6. *Let (A, π) be a topologically trivial lattice. Then the interval topology on (A, π) is the coarsest topology.*

Proof. Let $[a)$ $(= \{x : x \geqq a\})$ be an infinite set for a given $a \in A$. By Theorem 5 $(a] = \{x : x \leqq a\} = \{a\}$. If $b \in A - [a)$, then $b \parallel a$ and $b \wedge a < a$ which is impossible. So $[a) = A$. Similarly for the sets of the form $(a]$. The sets of the form $(a]$, $[a)$ form a subbasis of the closed sets of interval topology. As all of them are finite or equal to A, interval topology is the coarsest topology.

Now we shall find a condition for (A, π) to be topologically discrete. We shall start with the following assertion.

Theorem 7. *Let (A, π) be a partially ordered set. Then the interval topology is the greatest element of $S(A, \pi)$ exactly when the following condition (P) is satisfied: for every two elements a, b $a \parallel b$ there exist a system $a_1, \ldots, a_n > a$ and a system $a'_1, \ldots, a'_m < a$ so that $[b) - \bigcup_{i=1}^{n} [a_i)$ and $(b] - \bigcup_{i=1}^{m} (a'_i]$ are finite.*

For proof see [4].

Corollary of the Theorem 7. (A, π) *is topologically discrete exactly when the interval topology on* A *is discrete and* (P) *holds.*

Proof is evident.

Remark. There exist partially ordered sets (A, π) even distributive lattices (see figure 1), in which the interval topology is discrete and (P) does not hold.

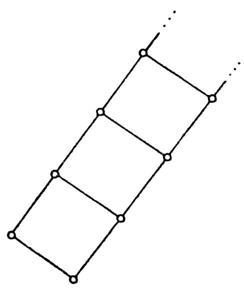

Fig. 1.

For the lattices we get from Theorem 7 the following

Theorem 8. *Interval topology on a lattice* (A, π) *is the greatest element of* $S(A, \pi)$ *exactly when for* $a \parallel b$, $[b) - [a \vee b)$ *and* $(a] - (a \wedge b]$ *are finite sets.*

Hence we get

Corollary of the Theorem 8. *In distributive lattice* (A, π) *with the least element (the greatest element) and with infinitely many (dual) atoms interval topology is not the greatest element in* $S(A, \pi)$.

Proof. Let a be the least element and a_i, $i \in I$ the atoms in (A, π). Let a_{i_1} and a_{i_2} be two different atoms and $j \in I$, $j \neq i_1$, $j \neq i_2$. Then $a_j \vee a_{i_2} \in [a_{i_2})$ and $a_j \vee a_{i_2}$ non $\in [a_{i_1} \vee a_{i_2})$. Simultaneously $a_j \vee a_{i_2} \neq a_{j'} \vee a_{i_2}$ for $j \neq j'$. Hence $[a_{i_2}) - [a_{i_1} \vee a_{i_2})$ is infinite.

The dual assertion can be proved in a similar way.

Some results on the existence of the greatest element or maximal ones in $S(A, \pi)$ for a given (A, π) can be found in [4]. The general solutions of the related problems are unknown. The questions concerning the existence of a topology with prescribed properties in $S(A, \pi)$ are mostly open, too.

We shall conclude giving a necessary condition for the existence of a compact topology in $S(A, \pi)$ for a lattice (A, π).

Theorem 9. *Let there exist in* $S(A, \pi)$ *for a lattice* (A, π) *a compact topology* $\boldsymbol{u}$. *Then* (A, π) *has the greatest and least elements.*

Proof. Admit that A has not the greatest element. For each element $a \in A$ an open set O_a exists in u such that $(a] - \{a\} \subset O_a \subset A - [a)$ (from the definition of compatibility). For every a one such O_a will be chosen. It follows from our assumption that the system $\{O_a : a \in A\}$ so constructed is an open covering of A in u. There exists a finite subcovering $O_{a_1}, \ldots, O_{a_n}$. Then we have $a_1 \vee \ldots \vee a_n$ non $\in O_{a_1} \cup \ldots \cup O_{a_n}$ which is impossible.

Similarly for the least element.

From the Theorem 5 and the example on fig. 2 it follows that Theorem 9 is not valid for the partially ordered sets in general. Further, let A be the lattice on fig. 3. Define a topology u on A by means of the following subbasis of open sets $\mathfrak{S} = \{\{a_k\}, \{b_l\}, \{c\} \cup \{a_k, a_{k+1}, \ldots, \ldots, b_{l-1}, b_l\} : k, l$ positive integer$\}$. u is compact and compatible with the order and A is not complete.

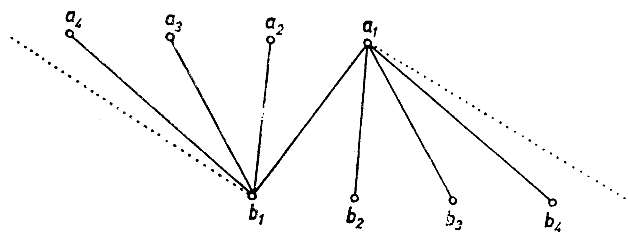

Fig. 2.

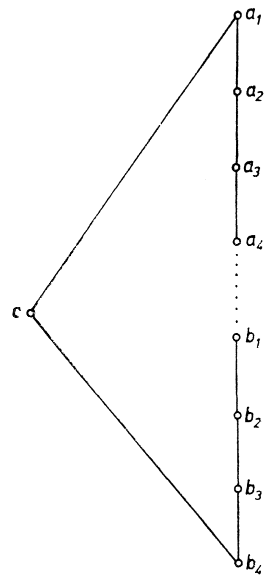

Fig. 3.

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