J. C. Abbott

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REMARKS ON AN ALGEBRAIC STRUCTURE FOR A TOPOLOGY

J. C. ABBOTT

Annapolis

The purpose of this paper is to discuss an algebraic structure closely related to topology. Thus a topology can be defined in terms of a carrier space, X, and a neighborhood mapping, η, which assigns a neighborhood filter to each point of X. The principal interest of the topologist is the space X and the ways in which the topology affects the structure of this space. In this paper we shall impose an algebraic structure on the topology itself and be more concerned with the algebraic behavior of this structure than with the structure of the space itself. We begin with a discussion of purely algebraic matters concerned with this algebraic system and later we shall discuss its relations with topology.

1. Implication Algebra. In previous papers we have defined an *implication algebra* as a *system*, $\langle I, .\rangle$, *consisting of a carrier set*, I, *together with a single binary operation*, ab, $a, b \in I$, *satisfying*:

$$\begin{aligned} &\text{P1:} \quad (ab)\,a = a\,; \\ &\text{P2:} \quad (ab)\,b = (ba)\,a\,; \\ &\text{P3:} \quad a(bc) = b(ac)\,. \end{aligned}$$

P1 – P3 imply that the product, aa, is a fixed element of I independent of a. We designate it by 1. This element is then simultaneously a left identity and right zero of I, i.e., $aa = 1$, $1a = a$, and $a1 = 1$ $\forall a \in I$. Furthermore, using only P1 – P3 we can introduce a partial order in I by $a \leqq b \Leftrightarrow ab = 1$. Under this partial order the element $a \vee b = (ab)\,b$ is a least upper bound for a and b, and 1 is a greatest element of I. Hence, $\langle I, \leqq \vee, 1\rangle$ is a union semi-lattice. I is not, in general, a lattice, since not every pair of elements determine a greatest lower bound within I. However if a and b have any lower bound, p, then the element $a \wedge b = [a(bp)]\,p$ is a greatest lower bound for a and b. Next, we call the set of left multiples of a given element, a, the *principal left ideal generated by a* and designate it by $[a]$. However P1 – P3 imply that $b = xa \Leftrightarrow a \leqq b$, so that this concept is identical with that of the principal filter generated by a in $\langle I, \leqq\rangle$. Since every principal ideal is bounded below, it follows that it is a lattice. In fact this lattice is both distributive and complemented, i.e., a boolean algebra. Thus if p is fixed and $a \in [p]$, then ap is a complement of a relative to p and 1, i.e., $a \vee ap = 1$ and $a \wedge ap = p$. Hence, every implication algebra is a union semi-lattice in which every principal ideal is a boolean algebra. The converse is also true, so that we have a purely lattice theoretic characterization of implication algebra. In fact, if $\langle I, \vee \rangle$ is a union semi-lattice in which every principal ideal is boolean, then the definition $ab = (a \vee b)'_b$ where x'_b denotes the complement of x within $[b]$, converts $\langle I, \vee \rangle$ into an implication algebra. As a corollary we can give a purely implication characterization of boolean algebra as an implication algebra satisfying:

$$\text{P4:} \quad \exists 0 \in I \quad \textit{such that} \quad 0a = 1 \quad \forall a \in I\,.$$

In this case $\langle I, ., 0 \rangle$ is equal to the principle ideal generated by 0 which is therefore boolean. Hence, a boolean algebra can be defined as a set, I, with a single binary operation and a single distinguished element (nullary operation) safisfying P1 – P4. Finally, we call I complete, if it is complete as a union semi-lattice.

Many of the most common examples of implication algebras come from set theory and topology. Thus if X is any set, then its power set, $\mathscr{P}(X)$, is a boolean algebra and therefore an implication algebra. Hence, any sub-implication algebra, i.e., subset closed under set implication $AB = A \to B = A' \cup B$ where $'$ denotes complement within X, is also an implication algebra. Moreover, the dual set operation, $AB = B - A = A' \cap B$ also satisfies P1 – P3 within $\mathscr{P}(X)$, so that $\mathscr{P}(X)$ is also an implication algebra under subtraction. We call $\langle \mathscr{P}(X), \to \rangle$ an *implication algebra* and $\langle \mathscr{P}(X), - \rangle$ a *subtraction algebra*. We then use the term "*semi-boolean algebra*" to designate either an implication algebra or a subtraction algebra. The only distinction between an implication algebra and a subtraction algebra lies in the definition of the partial order, i.e., in an implication algebra we write $a \leqq b \Leftrightarrow ab = 1$ whereas in a subtraction algebra $a \geqq b \Leftrightarrow ab = 0$, i.e., the two partial orders are converse to each other. Hence, as abstract algebras they are identical. Returning now to $\mathscr{P}(X)$, the set of all non-empty subsets of X is also an implication algebra which we designate by $\mathscr{P}^-(X)$. Since it does not have a least element, it is clearly not boolean (for $\overline{\overline{X}} \geqq 2$). Dually the set of proper subsets of X is a subtraction algebra. More generally, if α is the cardinality of X and $\beta \leqq \alpha$, then the set of subsets with cardinality $\geqq \beta$ $(> \beta)$ is an implication algebra while the set of subsets with cardinality $\leqq \beta$ $(< \beta)$ is a subtraction algebra. In particular, the set of subsets with finite complement is an implication algebra while the set of finite subsets is a subtraction algebra. Also the set of infinite subsets of any infinite set is a non-boolean implication algebra. Finally, if A is a fixed set, then the set of supersets of A (or proper supersets) is an implication algebra.

If now $\langle X, \mathfrak{T} \rangle$ is a topological space with topology $\mathfrak{T}$ then the prime example of an implication algebra connected with $\langle X, \mathfrak{T} \rangle$ is the set, $\mathfrak{N}$, of all neighborhoods within X, where N is a neighborhood, if it has a non-empty interior. For if A and B have non-empty interiors, then clearly $A' \cup B$ does. Again $\mathfrak{N}$ is not boolean, since the empty set is not a neighborhood.

We next investigate the theory of ideals, homomorphisms, and congruence relations on an abstract implication algebra. First, a subset, $H \neq \mathscr{I}$ of an implication algebra, I, is called an *implication ideal* if it satisfies: (i) $a \in H \Rightarrow xa \in H \; \forall x \in I$ and (ii) $a, b \in H$ and $\exists a \wedge b \Rightarrow a \wedge b \in H$. This is equivalent to the usual definition of a dual ideal in a lattice, except that the existence of the meet, $a \wedge b$, must be postulated. An *implication homomorphism* is next defined in the usual way as a mapping on I onto an implication algebra, $\bar{I}$, which preserves implication. Its *kernel* is the set of inverse images of the unit, $\bar{1}$, of $\bar{I}$. A *congruence relation*, $\equiv$, is an equivalence relation which has the left and right substitution properties with respect to implication. Every congruence relation determines a quotient algebra, $\bar{I}$, of cosets and the natural mapping, $a \to \bar{a} = \{x \mid x \equiv a\}$, is a homomorphism with kernel $\bar{1}$. Finally, every ideal, H, determines a congruence relation given by $a \equiv b \bmod H \Leftrightarrow \Leftrightarrow ab, ba \in H$. The cycle is now completed by showing that if Θ is a homomorphism from I onto $\bar{I}$ with kernel K, then $I \bmod K$ is isomorphic to $\bar{I}$. In the case I is a lattice, i.e., closed under meet, then the definition of congruence relation given here reduces to the usual lattice theoretic definition, $a \equiv b \bmod H \Leftrightarrow \exists h \in H$ such that $a \wedge h \equiv \equiv b \wedge h$.

We can now consider the structure of the family, $\mathscr{I}(I)$, of all ideals of an arbitrary implication algebra I. This family is clearly a complete lattice under set inclusion with least and greatest elements, the ideals $\Phi = [1]$ and I itself. The meet of two ideals, H and K, is their point set intersection, but their union is the smallest ideal containing them. We designate it by $H \sqcup K$ or $\sqcup H_\alpha$ for arbitrary unions. The set, $\mathscr{P}$, of principal ideals is dually isomorphic to I itself, so that this suggests defining an implication product for ideals corresponding to the product of two elements. Such a product can be generalized to arbitrary ideals and leads to the definition: *if H and K are two ideals of an implication algebra, I, then the set $HK = \{k \in K \mid hk = k \; \forall h \in H\}$ is called their implication product.* This definition is equivalent to $HK = \{k \in K \mid h \vee \vee k = 1 \; \forall h \in H\}$. If we consider the special case where $K = I$, then the ideal $H_* = = HI$ is characterized as the maximal ideal satisfying $H \cap X = \Phi$, i.e., H_* is a lower pseudo-complement of H within I. Hence, $\mathscr{I}$ is a lower pseudo-complemented lattice. More generally $HK = HI \cap K$ and the operation HK satisfies P1, P3, and P4 for an implication algebra, but P2 is replaced by the inequality: P2a: $H \cap K \subseteq (HK)K$ (we use $H \cap K$ instead of $H \sqcup K$ because of the duality of the isomorphism between I and $\mathscr{P}$). Within $\mathscr{I}$, HK can be characterized somewhat similarly to the characterization of implication in terms of union and complement in an implication algebra, i.e., $HK = (H \cap K)_{*K}$ where X_{*K} denotes the lower pseudo-complement of the ideal X within the principal ideal in $\mathscr{I}$ generated by K. Thus corresponding to the characterization of implication algebra in terms of union semi-lattices, we can characterize $\mathscr{I}$ as the dual of a distributive lattice in which every principal ideal is upper pseudo-complemented.

Although I is not a boolean algebra, it does contain two special subsets which are boolean algebras. First, the operation, $H \to H^- = H_{**}$ is a closure operation and an ideal is called closed if $H = H^-$. The class of all closed ideals, $\mathscr{F}$, is then a sub-implication algebra of $\mathcal{I}$ which is itself boolean (although not a sub-boolean algebra). Next, we call an ideal *complemented* if it satisfies $H \sqcup HI = I$, i.e., if H_* is a true complement. Again, if $\mathcal{C}$ is the class of complemented ideals, then $\mathcal{C}$ is also a boolean algebra. In general we have $\mathcal{P} \subseteq \mathcal{C} \subseteq \mathcal{F} \subseteq \mathcal{I}$. We mention one final result which is useful in developing a Zassenhaus lemma and Jordan-Holder theorem for implication algebra. Namely, if H is a complemented ideal, then $I \bmod H$ is isomorphic to HI and conversely, $I \bmod HI \cong H$.

2. Neighborhoods Algebras. We can now indicate some of the interrelations between implication algebra and topology. First of all topology serves as a rich source of examples for results and counterexamples in implication algebra. Thus if x is a point of a topological space $\langle X, \mathfrak{T}\rangle$, then the set, $\mathfrak{N}_x$, of all neighborhoods of x is a filter (i.e., ideal) in the implication algebra, $\mathfrak{N}$, of all neighborhoods of X. Hence, a topology determines a mapping, η, defined on points of X into the family of ideals of $\mathfrak{N}$. This mapping can be extended to all of $\mathcal{P}(X)$ by definining $\eta(A) = \mathfrak{N}_A$ to be set of all neighborhoods of A where A is any subset of X. Thus N is a neighborhood of $A \Leftrightarrow \exists G \in \mathfrak{T}$ such that $A \subseteq G \subseteq N$. Conversely, every topology on X arises from such a neighborhood map as is conventional in modern approaches to topology. If we impose an additional compatibility condition on η, so that the union of open sets will remain open, then we can consider η to be defined on $\mathcal{P}(X)$. The role of the space X itself then becomes somewhat subordinated and we can generalize many topological concepts to implication algebras with suitable neighborhood maps. Thus if I is a union complete implication algebra then we call a mapping $\eta : a \to \eta(a)$ from I into the algebras of ideals of I a *neighborhood map*, if it satisfies:

(i) $x \in \eta(a) \Leftrightarrow \exists g$ such that $\eta(g) = [g]$ and $a \leqq g \leqq x$;
(ii) $\eta(g_\alpha) = [g_\alpha] \Rightarrow \bigcap \eta(g_\alpha) \subseteq \eta(\bigvee g_\alpha)$;
(iii) if $0 \in I$, then $\eta(0) = I$.

We call g *η-open* if $\eta(g) = [g]$ and the system $\langle I, \eta\rangle$ a *neighborhood algebra.*

It follows from (i) that η is antitone, i.e., $a \leqq b \Rightarrow \eta(b) \subseteq \eta(a)$. Hence also $\eta(a) \subseteq [a]$ and $\eta(1) = [1]$. We can show that the set of open sets forms a topology in the usual sense.

The set $\mathfrak{T} = \{g \mid \eta(g) = [g]\}$ of open elements of I is closed under finite meet and arbitrary unions. This statement can be proved as follows:

(i) Let $g_1, g_2 \in \mathfrak{T}$ and let $\exists g_1 \wedge g_2$ in I. Then $g_1 \wedge g_2 \leqq g_1, g_2 \Rightarrow \eta(g_1), \eta(g_2) \subseteq \eta(g_1 \wedge g_2)$, whence $\eta(g_1) \sqcup \eta(g_2) = [g_1] \sqcup [g_2] = [g_1 \wedge g_2] \subseteq \eta(g_1 \wedge \wedge g_2)$. But $\eta(g_1 \wedge g_2) \subseteq [g_1 \wedge g_2]$, so that the equality, $\eta(g_1 \wedge g_2) = [g_1 \wedge g_2]$ follows.

(ii) Let $\eta(g_\alpha) = [g_\alpha]$. Then $g_\alpha \leqq \bigvee g_\alpha \Rightarrow \eta(\bigvee g_\alpha) \subseteq \eta(g_\alpha)\ \forall\alpha$ and hence $\eta(\bigvee g_\alpha) \subseteq \subseteq \bigcap \eta(g_\alpha)$. But the reverse inequality follows from (ii) in the definition. Hence $\eta(\bigvee g_\alpha) = = \bigcap \eta(g_\alpha) = \bigcap [g_\alpha] = [\bigvee g_\alpha]$. Hence $\bigvee g_\alpha$ is open.

The condition (ii) is stated as an inequality for open elements. In fact, we can show that it is an equality for arbitrary elements, i.e., $\eta(\bigvee a_\alpha) = \bigcap \eta(a_\alpha)$. For $x \in \bigcap \eta(a_\alpha) \Rightarrow x \in \eta(a_\alpha)\ \forall\alpha$. By (i) $\exists g_\alpha$ open such that $a_\alpha \leqq g_\alpha \leqq x$. Hence $\bigvee a_\alpha \leqq \bigvee g_\alpha \leqq$

$\leqq x$, where $\bigvee g_\alpha$ is open. Therefore $x \in \eta(\bigvee a_\alpha)$ whence $\bigcap \eta(a_\alpha) \subseteq \eta(\bigvee a_\alpha)$ and the reverse inequality follows form the antitone law.

As an example, if X is any set then we can take $I = \mathscr{P}(X)$. The discrete topology is then defined by the neighborhood map $\eta(x) = [\{x\}]$ for $x \in X$ and $\eta(A) = [A]$ for any subset. On the other hand the trivial topology is given by $\eta(A) = [1]$ for $A \neq \emptyset$ and $\eta(\emptyset) = I$. On the other hand if H is a fixed ideal in $\mathscr{P}(X)$, then $\eta(A) = H \cap [A]$ is a neighborhood map. A set is then open if and only if it belongs to H. For example, if X is an infinite set of cardinality α and $\aleph_0 \leqq \beta \leqq \alpha$, then we can take as H the class of subsets of X whose complements have cardinality $< \beta$. If $\beta = \aleph_0$ we obtain the usual finite complement topology. On the other hand if A is a fixed set, then we can take $H = [A]$, the principal ideal generated by A. Open sets are sets which contain A. This topology fails to separate points within A but separates points outside of A. Again, if $\langle X, \mathfrak{T} \rangle$ is a topological space with closure operation $\bar{A}$, then we can define a neighborhood map by $\eta(A) = \{N \subseteq X \mid \exists G \in \mathfrak{T} \text{ and } \bar{A} \subseteq G \subseteq N\}$. The η-open sets are then the closed-open sets of $\langle X, \mathfrak{T} \rangle$.

Since $H = \eta(x)$ is an ideal in $I = \mathscr{P}(X)$, we can examine the quotient algebra, $\bar{I} = I \bmod H$. Thus, if A and B are two subsets of X, then $A \equiv B \bmod H \Leftrightarrow \exists$ a neighborhood, N_x, of x such that $A \cap N_x = B \cap N_x$. But $N_x \in H$ implies $\exists$ an open neighborhood, G_x such that $A \cap G_x \equiv B \cap G_x$. We therefore say that A and B behave alike locally. The quotient structure therefore describes the local behavior of X at the point x. This example also serves to illustrate various aspects of the theory of implication algebras. For example, since union is defined in terms of implication, every implication homomorphism preserves finite unions. On the other hand they do not necessarily preserve arbitrary unions. Thus, for example, let $\langle X, \mathfrak{T} \rangle$ be a, say, metric space with metric d. Then if H is again the neighborhood filter of a point a, we can let θ be the natural homomorphism from $I = \mathscr{P}(X)$ onto $I \bmod H$. Let $A_n = \{x \in X \mid d(x, a) > 1/n\}$. Then θ maps A_n onto the fixed coset $\bar{A} = \{Y \subseteq X \mid d(Y, a) \neq 0\}$. Hence $\bigcup \bar{A}_n = \bar{A}$. On the other hand $\overline{\bigcup A_n} = \{x\}'$ where $'$ denotes ordinary set complement. Thus $\overline{\bigcup A_n} \neq \bigcup \bar{A}_n$.

As final example, we use I and H as described here to illustrate the product of ideals and pseudo-complements. Thus $HI = \{Y \subseteq X \mid Y \cup N_x = X \ \forall N_x \in H\}$. But $Y \cup N_x = X \Leftrightarrow N_x' \subseteq Y$. Hence, either $Y = X$ or $Y = \{x\}'$, i.e., $HI = H_*$ contains exactly two elements and is an atom in $\mathscr{I}$. The closure $H^- = (HI)\, I$ is then $\{Z \mid Z \cup Y = X \ \forall Y \in HI\}$. Hence, $H^- = [\{x\}]$ is the principal filter generated by singleton $\{x\}$. Thus, in this case, $(HI)\, I \subset H$. H is therefore neither a closed nor a complemented ideal and $I \bmod H \not\cong HI$. Similarly, $(HK)\, K \neq H \cap K$ and P2 fails in I.

Various topological concepts can now be introduced into neighborhood algebras without direct reference to an underlying space X. For example the concept of a filter basis, an ultra filter, continuous mapping, homeomorphism, etc. can be defined.

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