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ON THE LATTICE OF TOPOLOGIES

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The family Σ of all topologies on a set X forms a lattice under the partial ordering of inclusion. The largest element, 1, is the discrete topology and the smallest element, 0, is the trivial topology. Since the intersection of any family of topologies on X is again a topology on X, Σ is a complete lattice.

The topologies of the form $\{X, E, \emptyset\}$ for $\emptyset \neq E \neq X$ are the atoms and every topology on X, different from 0, is the supremum of atoms.

The topologies $\tau(x, \mathscr{U}) = \{E \subset X : x \notin E \text{ or } E \in \mathscr{U}\}$, where $x \in X$ and $\mathscr{U}$ is an ultrafilter different from the fixed ultrafilter at x, are co-atoms of Σ and in 1964, O. Fröhlich [12] showed that co-atoms also generate Σ. The co-atoms, called *ultraspaces* by Fröhlich, fall into two classes: the *principal ultraspaces*, when $\mathscr{U}$ is a principal (or fixed) ultrafilter on X, and the *nonprincipal* (or T_1) *ultraspaces*, when $\mathscr{U}$ is a nonprincipal (or free) ultrafilter.

The nonprincipal ultraspaces generate a complete sublattice Λ of Σ which is the lattice of T_1-topologies. Λ has 1 as the finest element and the cofinite topology, $\mathscr{C}$ as the coarsest.

The principal ultraspaces generate a sublattice Π of Σ of *principal topologies*. The elements of Π are characterized by the property that arbitrary intersections of open sets are open. Π is a complete lattice sharing the largest and smallest elements with Σ. Π is a meet-complete sublattice of Σ but is not a complete sublattice since the atoms of Σ are in Π [27].

A topology on X which is neither T_1 nor principal is a *mixed topology*. A mixed topology can be represented as the infimum of a T_1-topology and a principal topology, but this representation need not be unique. The supremum of two mixed topologies can be either T_1, principal, or mixed; the infimum however, can never be T_1. Other than this, not much is known about mixed topologies.

In 1958, Hartmanis [17] proved that the lattice of topologies on a finite set is complemented. Since every topology on a finite set is principal, $\Sigma = \Pi$. From the fact that Π is generated by principal ultraspaces, a simple proof shows that Π is always complemented, regardless of the cardinality of the set X.

Theorem 1. Π *is a complemented lattice.*

Proof. For $\tau \in \Pi$, let $\mathcal{D}$ be the decomposition of X defined by: $x, y \in D \in \mathcal{D}$ if and only if either $\tau \leqq \tau(x, \mathcal{U}(y))$ or $\tau \leqq \tau(y, \mathcal{U}(x))$. Choose one element x_D from each $D \in \mathcal{D}$. If $\tau_1 = \bigwedge\{\tau(x_D, \mathcal{U}(x_E)) : D, E \in \mathcal{D}\}$ and τ_2 is the infimum of all ultraspaces $\tau(x, \mathcal{U}(y))$ for which $\tau \leqq \tau(y, \mathcal{U}(x))$ but $\tau \nleqq \tau(x, \mathcal{U}(y))$, then $\tau_1 \wedge \tau_2$ is a complement for τ in Π.

In 1961, Gaifman [13], [14], showed that Σ is complemented if X is a countable set. To obtain this result, he proved that if every T_1-topology on a set has a complement, then every topology has one.

In 1966, the author [27] generalized Gaifman's result, and with the following theorems proved that Σ is a complemented lattice.

Theorem 2. *If every T_1-topology on a set X has a principal complement (i.e., a complement lying in Π), then every topology on X has a principal complement.*

Theorem 3. *If every T_1-topology with no isolated points has a principal complement, then every T_1-topology has a principal complement.*

Theorem 4. *A T_1-topology with no isolated points has a principal complement.*

Theorem 5. *The lattice Σ is complemented. Moreover, each topology has a principal complement.*

In 1968, Van Rooij [31] independently proved that Σ is complemented, using ideas similar to those in the proof of Theorem 4. His work did not depend upon the ultraspaces, nor upon Gaifman's work. Briefly, Van Rooij first proved that if a topological space can be well-ordered so that initial segments are closed, then the topology has a complement. Then, if τ is any topology on X, X can be inductively decomposed into a well-ordered sequence of subsets $\{X_\alpha\}_{\alpha<\gamma}$, so that for each $\alpha < \gamma$, X_α is dense in $\bigcup\{X_\beta : \alpha \leqq \beta < \gamma\}$ and each X_α is itself a well-ordered set with initial segments closed in the induced topology $\tau|X_\alpha$. Each $\tau|X_\alpha$ has a complement and these are used to define a complement for τ. In going through the construction of the complements, it is easy to see that they are all principal topologies.

Complementation in Σ is by no means unique, as has been shown by P. Schnare [24], [25]. He proved

Theorem 6 (Schnare [25]). *Every proper topology on an infinite set X has at least $|X|$ complements and at most $2^{2^{|X|}}$ complements ($2^{|X|}$ principal complements). Moreover, these bounds are the best possible.*

The lattice Λ of T_1-topologies is not complemented as can be seen from a very simple counterexample [28].

Example. Let (X_1, τ_1) be an infinite set with the cofinite topology, (X_2, τ_2) an infinite set with the discrete topology and let (X, τ) be the topological sum of X_1 and X_2. Assume τ has a complement τ' in Λ. For each $x \in X$, $\{x\} \in \tau \vee \tau'$. If $\{x\} \in \tau'$ for all $x \in X_1$, then $X_1 \in \tau \wedge \tau'$, but X_1 is not cofinite. Thus there must be an $x \in X_1$ such that $\{x\} \notin \tau'$. But $\{x\} = U \cap V$ where $U \in \tau'$ and $V \in \tau$. Since $X_1 - V$ is finite, $U \cap X_1$ must be finite. But τ' is a T_1-topology, so there is a $U^* \in \tau'$ such that $\emptyset \neq U^* \subset \subset U \cap X_2$. Thus, $U^* \in \tau \wedge \tau'$ but is not cofinite. Thus, if $\tau \vee \tau' = 1$, $\tau \wedge \tau' \neq \mathscr{C}$.

However, many T_1-topologies do have complements in Λ: the hyperplanes of Bagley [4]; the nonprincipal ultraspaces and their finite intersections; the order topology on a well-ordered set, [29]; and the usual topology on the reals [30].

B. A. Anderson [1], [2], [3] has used the technique of the author and E. F. Steiner for providing a complement for the reals, and has found large classes of T_1-topologies with T_1-complements. He also has some bounds on the number of T_1-complements.

Theorem 7 (Anderson [1]). *If X is an infinite set, there is a family $L \subset \Lambda$ such that $|X| \leqq L \leqq 2^{|X|}$ and any two elements of L are complementary.*

The same result also holds for Σ and Π.

If τ and τ' are T_1-complements, knowledge about τ gives almost none about τ', as Anderson has shown.

Theorem 8 (Anderson [1]). *Every set of cardinal c has a T_1-topology τ such that for any T_1-topology σ on a set S of cardinality c, τ has a T_1-complement with a subspace homeomorphic to (S, σ). An analogous statement holds in Σ.*

No characterization has as yet been given for those topologies in Λ which do not have T_1-complements.

Ultraspaces may be studied easily because of their point-ultrafilter representation. Many topological properties of ultraspaces have been studied by the author [27] and J. Girhiny [15].

Theorem 9. *For an ultraspace $\tau(x, \mathscr{U})$ the following are equivalent:*

(a) *$\mathscr{U}$ is nonprincipal,*
(b) *$\tau(x, \mathscr{U})$ satisfies T_1 to T_5,*
(c) *$\tau(x, \mathscr{U})$ is totally disconnected,*
(d) *$\tau(x, \mathscr{U})$ is zero-dimensional,*
(e) *$\tau(x, \mathscr{U})$ is regular (completely regular).*

Theorem 10. *For any ultraspace $\tau(x, \mathscr{U})$, the following are equivalent:*

(a) *$\mathscr{U}$ is principal,*
(b) *$\tau(x, \mathscr{U})$ is locally compact,*
(c) *$\tau(x, \mathscr{U})$ is locally connected,*
(d) *$\tau(x, \mathscr{U})$ satisfies the first axiom of countability.*

Those topologies in Σ which are maximal P or minimal P, for various topological properties P, have frequently been studied [5], [7], [9], [10], [18], [20], [21], [22], [23], [26] and many have been characterized. Most of these characterizations involve the representation of the space as an infimum of ultraspaces.

Theorem 11 ([27]). *A space is maximal regular if and only if it is a non-principal ultraspace or is of the form* $\tau(x, \mathcal{U}(y)) \wedge \tau(y, \mathcal{U}(x))$ *for* $x \neq y$.

A comprehensive list has been given by J. Girhiny [15] for twenty-seven such properties. Two interesting results of his are:

Theorem 12 (Girhiny [15]). *There are no maximal second countable spaces.*

Theorem 13 (Girhiny [15]). *If* τ *is first countable and* $\tau < \tau(x, \mathcal{U})$ *with* $\mathcal{U}$ *nonprincipal, then there is a* τ' *which is first countable and* $\tau < \tau' < \tau(x, \mathcal{U})$.

The characterizations of maximal locally compact, maximal connected, and minimal totally disconnected topologies are not known.

Girhiny [16] has also investigated when a topological property is preserved by the lattice operations of coarser and finer, and by finite and infinite meets and joins.

Although not purely a lattice problem, the question arises as to when the property of a space being minimal P is preserved under products. Since minimal $T_{31/2}$ (T_4) spaces are compact Hausdorff [6], minimality is always preserved by products. That the product of minimal Hausdorff spaces is minimal Hausdorff has been proved in [11], [18], and [19]. The question as to whether the product of minimal T_3 spaces is minimal T_3 has not been answered.

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