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1. Introduction

Recently various authors have introduced and studied various generalizations of stratifiable spaces, whose definitions appear in the next section. In particular, we have the (*completely*) *monotonically normal* spaces of Zenor, the *elastic* spaces of Tamano and Vaughan, and the *linearly stratifiable* spaces of Vaughan. In attempting to decipher the common denominator of these spaces we will introduce one more class of spaces, which we shall call *stratonormal* spaces. We will also state and prove a characterization of monotonically normal spaces which will play a critical role in the following results, the proofs of which will appear elsewhere:

(a) The class of monotonically normal spaces equals the class of completely monotonically normal spaces (this answers questions of Heath, Lutzer and Zenor).

(b) "Linearly stratifiable" implies "elastic" implies "stratonormal" implies "monotonically normal".

2. Preliminary results

For the sake of completeness, let us begin with the definitions of elastic and stratonormal spaces, linearly stratifiable spaces and monotonically normal spaces.

Definition 2.1 (see [3]). Let X be a space. Then

(a) Let $\mathscr{U}$ be any collection of subsets of X and let $\mathscr{R}$ be a relation on $\mathscr{U}$ (i.e., $\mathscr{R} \subset \mathscr{U} \times \mathscr{U}$). (We shall often write $U \leqq V$ instead of $(U, V) \in \mathscr{R}$. The relation $\mathscr{R}$ is said to be a *framing of* $\mathscr{U}$ provided that, for every $U, V \in \mathscr{U}$ with $U \cap V \neq \emptyset$, either $U \leqq V$ or $V \leqq U$.

(b) A collection $\mathscr{U}$ is said to be *framed in a collection* $\mathscr{V}$ *with frame map* f: $\mathscr{U} \to \mathscr{V}$ provided that there exists a framing $\mathscr{R}$ of $\mathscr{U}$ such that for every $\mathscr{U}' \subset \mathscr{U}$ which has an $\mathscr{R}$-upper bound we get that $\overline{\bigcup \mathscr{U}'} \subset \bigcup f(\mathscr{U}')$.

(c) If $\mathscr{U}$ is framed in $\mathscr{V}$ and $\mathscr{R}$ is a transitive relation then $\mathscr{U}$ is said to be *elastic* in $\mathscr{V}$.

(d) A pair-base $\mathscr{P}$ for X (i.e., $\mathscr{P}$ is a collection of pairs $P = (P_1, P_2)$ of subsets of X such that P_1 is open, $P_1 \subset P_2$ and, for each $x \in X$ and neighborhood U of x, there exists $(P_1, P_2) \in \mathscr{P}$ with $x \in P_1 \subset P_2 \subset U$) is said to be an *elastic* (*framed*) *base* if there is a framing of $\mathscr{P}_1 = \{P_1 \mid (P_1, P_2) \in \mathscr{P}\}$ such that $\mathscr{P}_1$ is elastic (framed) in $\mathscr{P}_2 = \{P_2 \mid (P_1, P_2) \in \mathscr{P}\}$ with respect to the map $f : \mathscr{P}_1 \to \mathscr{P}_2$ defined by $f(P_1) = P_2$.

(e) A T_1-space with an elastic (framed) base is called an *elastic* (*stratonormal*) *space*.

Definition 2.2 (see [4]). A T_1-space X is said to be *linearly stratifiable* provided that there exists some infinite cardinal number α (we assume that cardinal numbers are ordinal numbers) such that to each open $U \subset X$ one can assign a family $\{U_\beta\}_{\beta<\alpha}$ of open subsets of X such that

(a) $U_\beta^- \subset U$ for all $\beta < \alpha$,
(b) $\bigcup\{U_\beta \mid \beta < \alpha\} = U$,
(c) $U_\beta \subset V_\beta$ whenever $U < V$,
(d) $U_\gamma \subset U_\beta$ whenever $\gamma < \beta < \alpha$.

Definition 2.3 (see [2]). For any space X, let $\mathscr{D}_X = \{(A, B) \mid A$ and B are disjoint closed subsets of $X\}$, $\mathscr{S}_X = \{(A, B) \mid A, B \subset X$ and $A \cap \bar{B} = \emptyset = B \cap A^-\}$. The T_1-space X is said to be *monotonically normal* (respectively *completely monotonically normal*) provided that to each $(A, B) \in \mathscr{D}_X$ (respectively $(A, B) \in \mathscr{S}_X$) one can assign an open subset $G(A, B)$ of X such that

(a) $A \subset G(A, B) \subset G(A, B)^- \subset X - B$,
(b) $G(A, B) \subset G(A', B')$ whenever $A \subset A'$ and $B' \subset B$.

The function G is called a *monotone normality operator.*

We will now state and prove various characterizations of monotonically normal spaces. The techniques developed in the following result permit rather elementary proofs of most results announced in the introduction.

Theorem 2.4. *The following are equivalent*[1]):

(a) *X is completely monotonically normal.*
(b) *X is monotonically normal.*
(c) *To each pair (A, U) of subsets of X, with A closed, U open and $A \subset U$, we can assign an open set U_A such that*
(i) *$U_A \subset V_B$ whenever $A \subset B$ and $U \subset V$,*
(ii) *$U_A \cap (X - A)_{X-U} = \emptyset$.*

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[1]) P. Zenor informed us at the Third Prague Topological Symposium that R. Heath and D. Lutzer have also obtained the equivalence of (a) and (b). This has been confirmed by a letter from D. Lutzer, without proofs.

(d) *For each open* $U \subset X$ *and* $x \in U$ *there exists an open neighborhood* U_x *of* x *such that* $U_x \cap V_y \neq \emptyset$ *implies* $x \in V$ *or* $y \in U$.

(e) *There exists a base* $\mathscr{B}$ *for* X *such that, for each* $B \in \mathscr{B}$ *and* $x \in B$ *there exists an open neighborhood* B_x *of* x *such that*

$$B_x \cap C_y \neq \emptyset \quad \textit{implies} \quad x \in C \ \textit{or} \ y \in B.$$

Proof. Clearly (a) implies (b). Therefore we first prove that (b) implies (c): By Lemma 5.1 of [2], X has a monotone normality operator G such that $G(A, B) \cap \cap G(B, A) = \emptyset$ for each $(A, B) \in \mathscr{D}_X$ (indeed, the proof consists of observing that, for any monotone normality operator H on X, letting $G(A, B) = H(A, B) - H(B, A)^-$ will do the trick). Now, for each pair (A, U) with A closed, U open and $A \subset U$, let

$$U_A = G(A, X - U).$$

It is quite easy to see that

(i) $U_A \subset V_B$ whenever $A \subset B$ and $U \subset V$,
(ii) $U_A \cap (X - A)_{X-U} = \emptyset$.

Next we prove (c) implies (d): For each open $U \subset X$ and $x \in U$, let

$$U_x = U_{\{x\}}.$$

Suppose $U_x \cap V_y \neq \emptyset$, $x \notin V$ and $y \notin U$. Then $U_x \cap (X - \{x\})_{X-U} = \emptyset$ and $V_y \subset \subset (X - \{x\})_{X-U}$. Consequently $U_x \cap V_y = \emptyset$, a contradiction. This proves that $U_x \cap V_y \neq \emptyset$ implies $y \in U$ or $x \in V$.

Clearly (d) implies (e). So we complete the proof by showing that (e) implies (a): For each $(A, B) \in \mathscr{S}_X$, let

$$G(A, B) = \bigcup\{U_x \mid x \in A,\ U \subset X - B,\ U \in \mathscr{B}\}.$$

Clearly $G(A, B) \subset G(A', B')$ whenever $A \subset A'$, $B' \subset B$ and (A, B), $(A', B') \in \mathscr{S}_X$; each $G(A, B)$ is open and $A \subset G(A, B)$. Therefore we only need to prove that $G(A, B)^- \subset X - B$. Actually we prove the stronger result that $G(A, B) \cap G(B, A) = \emptyset$: Assume there exists $w \in G(A, B) \cap G(B, A)$. Then $w \in U_x$ for some $x \in A$ and some $U \subset X - B$ $(U \in \mathscr{B})$ and $w \in V_y$ for some $y \in B$ and some $V \subset X - A$ $(V \in \mathscr{B})$. Then $U_x \cap V_y \neq \emptyset$ which implies that $y \in U$ or $x \in V$, a contradiction (for example, $y \in U \subset X - B$ contradicts the fact that $y \in B$).

3. Results and questions

Among the various results we have so far obtained we mention the following:

1. *Every elastic space is monotonically normal* (indeed we have constructed a monotone normality operator for elastic spaces which satisfies much stronger properties than those in Theorem 2.4 (c)).

2. *The closed continuous image of a monotonically normal space is monotonically normal.* (The proof depends on Theorem 2.4 (d).)

3. *A monotonically normal space X is paracompact if and only if each open cover of X has a* (not necessarily open) *σ-cushioned refinement.*

4. Let X be monotonically normal. *To each pair (A, U) of subsets of X, with A closed, U open and $A \subset U$, one can assign a continuous function $f_{U,A}: X \to I$ such that $f_{U,A}(A) = 0$, $f_{U,A}(X - U) = 1$ and $f_{U,A} \geqq f_{V,B}$ whenever $A \subset B$ and $U \subset V$.* (The proof is essentially the same as the proof of Theorem 5.1 of [1], because of Theorem 2.4 (c).)

5. *Every monotonically normal space is hereditarily monotonically normal.* (This is immediate from Theorem 2.4 (d) and answers a question of Heath and Lutzer.)

6. (Theorem 3.3 of [2].) *Any linearly ordered topological space $(X, \leqq, \tau)$ is completely monotonically normal.* (While the original proof of this result by Heath and Lutzer is quite long, our Theorem 2.4 (e) allows the following very elementary proof: Without loss of generality, we assume that X has no $\leqq$-first and no $\leqq$-last point (if necessary, add rays and apply 5.). Let $\mathscr{B}$ be the base for X which consists of all open intervals and well-order X by the order W. For each $B \in \mathscr{B}$ and $x \in B$, let

$$B_x =]l(x, B), r(x, B)[$$

where $l(x, B)$ $(r(x, B))$ is the W-first element of B such that $l(x, B) < x$ $(x < r(x, B))$. It is quite easy to see that the B_x just defined satisfy Theorem 2.4 (e).)

The following two questions appear intriguing: Is the closed continuous image of an elastic (stratonormal) space an elastic (stratonormal) space? (Tamano and Vaughan [3] conjecture a positive answer, for the first case.)

References

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[4] *J. E. Vaughan:* Linearly stratifiable spaces. Pacific J. Math. (to appear).

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