ONE RESULT ON INVERSE LIMITS AND HYPERSPACES

R. DUDA

Wrocław

Hyperspaces, i.e. families consisting of non-empty subsets of a given space X and provided with a topology connected with that of X, are becoming nowadays a subject of intense research. In the present note we restrict our attention to compact Hausdorff spaces only and by a hyperspace we shall mean any family consisting of compact Hausdorff subspaces of a given compact Hausdorff space X. Our aim is to show that some "hyperfunctors" transforming compact Hausdorff spaces into some of its hyperspaces commute with the inverse limits, i.e.,

$$\mathfrak{H}(\lim \boldsymbol{X}) = \lim \mathfrak{H}(\boldsymbol{X}),$$

where $\mathfrak{H}$ is a hyperfunctor and $\boldsymbol{X}$ is an inverse system.

Let $\mathfrak{A}$ be the category consisting of all compact Hausdorff spaces and all continuous mappings, and let $\mathfrak{B}$ be a subcategory of $\mathfrak{A}$. A covariant functor $\mathfrak{H} : \mathfrak{B} \to \mathfrak{A}$ will be called a *hyperfunctor* of $\mathfrak{B}$ if, for each $(f : X \to Y) \in \mathfrak{B}$, $\mathfrak{H}(X)$ is a hyperspace of X, $\mathfrak{H}(Y)$ is a hyperspace of Y, and the induced mapping $\mathfrak{H}(f) : \mathfrak{H}(X) \to \mathfrak{H}(Y)$ is defined by $(\mathfrak{H}(f))(A) = f[A]$ for each $A \in \mathfrak{H}(X)$.

A subcategory $\mathfrak{B}$ will be called *closed with respect to hyperfunctor* $\mathfrak{H}$ if, for each inverse system $\boldsymbol{X} = \{X_\alpha, \pi^\alpha_\beta, \Sigma\}$ in $\mathfrak{B}$, it satisfies the following two conditions:

(i) the limit of $\boldsymbol{X}$ does exist and belongs to $\mathfrak{B}$ together with all projections $\pi_\alpha : \lim \boldsymbol{X} \to X_\alpha$,

(ii) for each "partial" inverse system $\boldsymbol{A} = \{A_\alpha, \pi^\alpha_\beta \mid A_\alpha, \Sigma\}$ such that $A_\alpha \in \mathfrak{H}(X_\alpha)$ and $\pi^\alpha_\beta \mid A_\alpha : A_\alpha \to A_\beta$ are onto, the limit of $\boldsymbol{A}$ does exist and belongs to $\mathfrak{H}(\lim \boldsymbol{X})$.

Some of the well known examples of hyperfunctors are Comp, C and Conv. If X is a compact Hausdorff space, then Comp (X) consists of all non-empty compact subsets of X, $C(X)$ consists of all non-empty compact and connected subsets (i.e., of all subcontinua) of X, and in the case of X being metric, Conv (X) consists of all non-empty compact convex subsets of X. If the topology in hyperspaces is that of Vietoris[1]), it is also known that the category $\mathfrak{B}_1 = \mathfrak{A}$ is closed with respect to Comp

[1]) A base of the Vietoris topology in a hyperspace $\mathfrak{H}(X)$ consists of all sets of the form

$$\langle U_1, \ldots, U_n; \mathfrak{H}(X)\rangle = \{A \in \mathfrak{H}(X) : A \subset \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \emptyset \text{ for } i = 1, 2, \ldots, n\},$$

where $U_1, \ldots, U_n$ are open in X and n is any integer.

(cf. [4], Propositions 4.9.2 and 5.10.1, and [3], Theorem 3.2.10), the full subcategory $\mathfrak{B}_2$ of all continua is closed with respect to C (cf. [3], Theorem 3.1.5), and the subcategory $\mathfrak{B}_3$ of all convex subcontinua of the Hilbert cube and mappings preserving convexity[2]) is closed with respect to Conv (cf. [1], Theorem 5.1 (d), and [2], Corollary 1).

Theorem. *Let $\mathfrak{B}$ be a subcategory of $\mathfrak{A}$, closed with the respect to a hyperfunctor $\mathfrak{H}$. If*

$$X = \{X_\alpha, \pi^\alpha_\beta, \Sigma\} \tag{1}$$

is an inverse system in $\mathfrak{B}$, then

$$\mathfrak{H}(X) = \{\mathfrak{H}(X_\alpha), \mathfrak{H}(\pi^\alpha_\beta), \Sigma\} \tag{2}$$

is an inverse system in $\mathfrak{A}$ and

$$\mathfrak{H}(\lim X) = \lim \mathfrak{H}(X). \tag{3}$$

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathfrak{H}(\lim X) \arrow[dl, "\mathfrak{H}(\pi_\beta)"'] \arrow[d, "\mathfrak{H}(\pi_\alpha)"] \\
\mathfrak{H}(X_\beta) & \mathfrak{H}(X_\alpha) \arrow[l]
\end{tikzcd}$$

obtained by imposing the functor $\mathfrak{H}$ upon the system (1) augmented with its limit. Since $\mathfrak{H}$ is a functor, the diagram commutes and so (2) is an inverse system.

Since $\mathfrak{H}$ is a hyperfunctor, (2) is an inverse system in the category $\mathfrak{A}$. Hence its limit $\lim \mathfrak{H}(X)$ exists (cf. [3], Theorem 3.2.10) and so there exists a unique continuous mapping

$$h : \mathfrak{H}(\lim X) \to \lim \mathfrak{H}(X)$$

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[2]) A continuous mapping $f: X \to Y$ from a metric space X onto a metric space Y *preserves convexity* if it satisfies the following two conditions:

(i) if $K \subset X$ is a segment, then $f[K]$ is a segment or a point,
(ii) if $L \subset Y$ is a segment or a point, then $f^{-1}[L]$ is convex.

such that the diagram

$$\begin{tikzcd}
 & & \mathfrak{H}(\lim X) \arrow[dll, "\mathfrak{H}(\pi_\beta)"'] \arrow[dl, "\mathfrak{H}(\pi_\alpha)"] \arrow[dd, dashed, "h"] \\
\mathfrak{H}(X_\beta) & \mathfrak{H}(X_\alpha) \arrow[l, "\mathfrak{H}(\pi_\beta^\alpha)"'] & \\
 & & \lim \mathfrak{H}(X) \arrow[ull, "p_\beta"] \arrow[ul, "p_\alpha"']
\end{tikzcd}$$

commutes, p_α and p_β being projections.

To see what h is like, take $A \in \mathfrak{H}(\lim X)$ and put $A_\alpha = [\mathfrak{H}(\pi_\alpha)](A)$ for each $\alpha \in \Sigma$. Since $[\mathfrak{H}(\pi_\beta^\alpha)](A_\alpha) = [\mathfrak{H}(\pi_\beta)](A) = A_\beta$, the set $\{A_\alpha\}_{\alpha\in\Sigma}$ is a thread of the system (2), and since $p_\alpha h(A) = [\mathfrak{H}(\pi_\alpha)](A) = A_\alpha$, we have $h(A) = \{A_\alpha\}_{\alpha\in\Sigma}$.

Now we show that h is one-to-one. Assume, on the contrary, that there are two sets $B, C \in \mathfrak{H}(\lim X)$ such that $B \setminus C \neq \emptyset$ and

$$(4) \qquad B_\alpha = C_\alpha \quad \text{for each} \quad \alpha \in \Sigma .$$

Since $B \setminus C \neq \emptyset$, there is a thread $\{b_\alpha\}_{\alpha\in\Sigma}$ which is in B and not in C. In view of the equality $[\mathfrak{H}(\pi_\alpha)](B) = \pi_\alpha[B]$ we have $b_\alpha \in B_\alpha$ for each $\alpha \in \Sigma$, which together with the assumption (4) implies that $b_\alpha \in C_\alpha$ for each $\alpha \in \Sigma$. And by virtue of the last relation there exists, for each $\gamma \in \Sigma$, a thread $t_\gamma = \{c_\alpha^\gamma\}_{\alpha\in\Sigma} \in C$ such that $c_\alpha^\gamma = b_\alpha$ for all $\alpha \leqq \gamma$. The set of threads t_γ is a Moore-Smith sequence convergent to $\{b_\alpha\}_{\alpha\in\Sigma}$. In view of the compactness of C, there must be $\{b_\alpha\}_{\alpha\in\Sigma} \in C$ – a contradiction.

Hence h, being defined on a compact space $\mathfrak{H}(\lim X)$ and being one-to-one and continuous, must be a homeomorphism.

It remains to show that h is onto. For that purpose take $A \in \lim \mathfrak{H}(X)$. Then $\{p_\alpha(A)\}_{\alpha\in\Sigma}$ is a thread of the inverse system (2) and, consequently, $\mathcal{A} = \{p_\alpha(A), \pi_\beta^\alpha \mid p_\alpha(A), \Sigma\}$ is an inverse system consisting of spaces $p_\alpha(A) \in \mathfrak{H}(X_\alpha)$ and mappings $\pi_\beta^\alpha \mid p_\alpha(A)$ being onto, and so, since $\mathfrak{B}$ is closed with respect to $\mathfrak{H}$, its limit $\lim \mathcal{A}$ belongs to $\mathfrak{H}(\lim X)$. Therefore $p_\alpha(A) = [\mathfrak{H}(\pi_\alpha)](\lim \mathcal{A}) = p_\alpha h(\lim \mathcal{A})$ for each $\alpha \in \Sigma$ and, consequently, $h(\lim \mathcal{A}) = A$.

Thus the proof of Theorem is completed.

Three examples of hyperfunctors Comp, C and Conv and of subcategories $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{B}_3$ yield three corollaries. The first two are known, since they have been proved directly by S. Sirota and J. Segal (however, the second for metric continua only). Recall that the topology in hyperspaces is here that of Vietoris.

Corollary 1 (cf. Sirota [6]). *If* $\{X_\alpha, \pi^\alpha_\beta, \Sigma\}$ *is an inverse system of compact Hausdorff spaces, then*

$$\mathrm{Comp}\,(\lim\{X_\alpha, \pi^\alpha_\beta, \Sigma\}) = \lim\{\mathrm{Comp}\,(X_\alpha), \mathrm{Comp}\,(\pi^\alpha_\beta), \Sigma\}\,.$$

Corollary 2 (cf. Segal [5]). *If* $\{X_\alpha, \pi^\alpha_\beta, \Sigma\}$ *is an inverse system of Hausdorff continua, then*

$$C(\lim\{X_\alpha, \pi^\alpha_\beta, \Sigma\}) = \lim\{C(X_\alpha), C(\pi^\alpha_\beta), \Sigma\}\,.$$

Corollary 3. *If* $\{X_n, f_n\}$ *is an inverse system of convex subcontinua* X_n *of the Hilbert cube and mappings* f_n *preserving convexity, then*

$$\mathrm{Conv}\,(\lim\{X_n, f_n\}) = \lim\{\mathrm{Conv}\,(X_n), \mathrm{Conv}\,(f_n)\}\,.$$

References

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