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In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 389--391.

Persistent URL: http://dml.cz/dmlcz/700784

ON THE SHAPE CLASSIFICATION OF MANIFOLD-LIKE CONTINUA

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In [4] S. Mardešić and the author classified all (n-sphere)-like continua as to their shape. It was shown that such continua are of the shape of a point, the n-sphere or the $(n - 1)$-fold suspension of a solenoid. Shape is a classification of compacta which is coarser than homotopy type but which coincides with it on ANR's. K. Borsuk also considered the relation between likeness and shape in [1] where he gave an example of two compacta which are like each other but are of different shape. The general question of the shape classification of manifold-like continua remains open. However, in this note some recent progress in the case of projective plane-like continua is presented.

A metric continuum X is said to be *like a compactum* Y or *Y-like* provided, for each $\varepsilon > 0$, there is a mapping $f_\varepsilon : X \to Y$ onto Y such that $\operatorname{diam} f_\varepsilon^{-1}(y) < \varepsilon$ for any $y \in Y$. The connection between likeness and shape can be made through the following result of Mardešić and the author [3].

Theorem 1. *If Y is a polyhedron and X is a Y-like continuum, then X is the inverse limit of an inverse sequence of copies of Y.*

It was shown by I. Lončar and S. Mardešić in [2] that Y need only be required to be an ANR in this theorem.

The reader is referred to [4] or to Mardešić's survey of the theory of shapes appearing in this volume for a description of the ANR-system approach to shape. Suffice it to say here, that we work in a category whose objects are inverse sequences of ANR's and whose morphisms are defined as in the case of mappings of inverse sequences except that one only requires commutativity up to homotopy. One defines a homotopy of morphisms and obtains a notion of homotopy type of ANR-sequences. Then one shows that for a compact metric space X this homotopy type is independent of the ANR-sequence associated with it in a particular way. So as far as shape is concerned one might as well use any inverse limit representation given by Theorem 1. Finally two compact metric spaces X and Y are said to be of the same *shape* if they have ANR-sequences associated with them of the same homotopy type.

We next describe a new class of projective plane-like spaces called solenoidal projective planes. These play the same role in the case of projective plane-like continua that solenoids did for (1-sphere)-like continua. Our description relies on P. Olum's analysis [5] of the twisted degree of mappings of the projective plane P into itself. A map $f : P \to P$ for which the induced homomorphism on the fundamental group of P into itself is an isomorphism, i.e.,

$$(1) \qquad f_{\#} : \Pi_1(P) \xrightarrow{\approx} \Pi_1(P)$$

(since $\Pi_1(P) = Z_2$ there is only one such isomorphism), is determined up to homotopy class by the absolute value of the twisted degree of f given by

$$(2) \qquad f^* : H^2(P, Z^t) \to H^2(P, Z^t) .$$

Here Z^t denotes the group of twisted integers and both cohomology groups are infinite cyclic. We assume that for each of the infinite cyclic groups in (2) one of the two possible generators is chosen. Then the homomorphism f^* carries the generator of the first group into some integral multiple of the generator of the second. This integer, which defines the homomorphism uniquely, is called the *twisted degree* of the mapping f. The twisted degree agrees with the usual degree on orientable manifolds. In the special case of mappings of the projective plane into itself for which (1) holds, only odd integers occur as twisted degrees. Moreover, in this case two such maps are homotopic if and only if the absolute values of their twisted degrees are equal.

Let $Q = (q_1, q_2, \ldots)$ be a sequence of odd primes. Then we define the inverse sequence $\{X_n, p_{nn+1}\}$ where each X_n is a copy of P and the map $p_{nn+1} : X_{n+1} \to X_n$ is determined as follows. Let S^1 be the unit circle in the complex plane and represent S^2 as $S^1 \times [-1, 1]$ with the appropriate identifications. Let $g : S^2 \to S^2$ be the map which sends (x, t) to (x^{q_n}, t) and let $h : S^2 \to P$ be the identification map which identifies (x, t) and $(-x, -t)$. Since q_n is odd, p_{nn+1} is well defined by $p_{nn+1}(h(x, t)) = h(g(x, t))$. We define the *solenoidal projective plane* P_Q to be the inverse limit of the inverse sequence just described. Two sequences of odd primes $Q = (q_1, q_2, \ldots)$ and $Q' = (q'_1, q'_2, \ldots)$ are said to be *equivalent*, $Q \sim Q'$, provided it is possible to delete a finite number of terms from each so that every prime occurs the same number of times in each of the deleted sequences.

Theorem 2. *Two solenoidal projective planes P_Q and $P_{Q'}$, are of the same shape iff $Q \sim Q'$.*

In analogy to the case of the 2-sphere we have the following result.

Theorem 3. *Every projective plane-like continuum X has the shape of a point, the projective plane or a solenoidal projective plane.*

To prove this theorem one first uses Theorem 1 to get X as the inverse limit of an inverse sequence of projective planes. Then one considers (case 1) if infinitely many of the bonding maps induce the zero homomorphism or (case 2) if they all induce an isomorphism on the fundamental group of P into itself. Case 1 has two subcases depending on whether or not infinitely many of the maps are null-homotopic. One can then show that either subcase implies that X has the shape of a point. Case 2 also has two subcases depending on whether or not all bonding maps with sufficiently large indices have twisted degree one. If they do then X is of the shape of P. If not then X is of the shape of some solenoidal projective plane. (Details of the proofs can be found in [6].)

References

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