Spiros P. Zervos
Lattices and topology

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 475--484.

Persistent URL: http://dml.cz/dmlcz/700780

LATTICES AND TOPOLOGY

S. P. ZERVOS

Athens

Abbreviations. Iff = if and only if; ▮ denotes the end of a proof.

Notation. $P(E)$ = the set of all subsets of the set E.

1. The inverse of a function between two complete lattices

Motivation. Let E and H be nonvoid sets, f the extension $P(E) \to P(H)$ of a single(multiple)-valued function $\sigma : E \to H$ and f^{-1} (f^{+}) the inverse (superior inverse, in the sense given in Berge [3], p. 26) of $f : P(H) \to P(E)$ (f^{+} reduces to f^{-1} for single-valued σ). The general properties of f^{-1}, in relation to those of f, being in the background of many facts in General Topology (and elsewhere), it seems interesting to search for an analogously useful definition of the "inverse" f^{-1} of a single-valued function f between two complete lattices P and Q.

Notation. X, X', X'' and X_i (Y, Y', Y'' and Y_j) denote elements of $P(Q)$; let f be a single-valued function $P \to Q$, $X_0(Y_0)$ the minimum and $X_u(Y_u)$ the maximum of $P(Q)$. Let (X_i) $[(Y_j)]$ be the family $(X_i)_{i \in I}[(Y_j)_{j \in J}]$. $\{X \mid \ldots\}$ = the set of all X such that.... Notice: In Bourbaki and in our previous papers, the above-mentioned $f^{-1} : P(H) \to P(E)$ was written with -1 above f.

Permanent hypothesis. $\forall Y$, $\{X \mid f(X) \leqq Y\} \neq \emptyset$. In particular, this is fulfilled if $f(X_0) = Y_0$.

Definition 1. $\forall Y$, $f^{-1}(Y) = \bigvee\{X \mid f(X) \leqq Y\}$.

Special case. For a binary relation R on $E \times H$, $X(\subseteq E) \mapsto R(X)$ defines a single-valued function $f : P(E) \to P(H)$. The f^{-1} just defined gives then what can be called "the superior inverse R^{+}" of R (distinct from what is usually denoted by R^{-1} and in Bourbaki by -1 above R); it reduces to f^{+} when R is a function.

Abbreviations. $f(\bigvee) \leqq \bigvee f$ stays for: $\forall (X_i)$, $f(\bigvee X_i) \leqq \bigvee f(X_i)$; $f^{-1}(\bigvee) \geqq \bigvee f^{-1}$ stays for: $\forall (Y_j)$, $f^{-1}(\bigvee Y_j) \geqq \bigvee f^{-1}(Y_j)$; and all analogous abbreviations.

An immediate consequence of Definition 1 is

Proposition 1. a) $\forall X,\ f^{-1}\circ f(X) \geqq X$;

b) f^{-1} *is isotone*; *hence also* $f^{-1}(\bigwedge) \leqq \bigwedge f^{-1}$ *and* $\bigvee f^{-1} \leqq f^{-1}(\bigvee)$;

c) $f^{-1}(Y_u) = X_u$.

Obvious remarks: $[\forall Y, f\circ f^{-1}(Y) = Y] \Rightarrow [f$ is surjective$]$; $[\forall X, f^{-1}\circ f(X) = X] \Rightarrow [f^{-1}$ is surjective and f is injective$]$; $[f$ is isotone$] \Rightarrow [f(X_0) = Y_0]$.

Proposition 2. *If* $f(\bigvee) \leqq \bigvee f$, *then* $\forall Y,\ f\circ f^{-1}(Y) \leqq Y$.

Proof. $f^{-1}(Y) = \bigvee X_i$, with $f(X_i) \leqq Y$. Hence

$$f\circ f^{-1}(Y) = f(\bigvee X_i) \leqq \bigvee f(X_i) \leqq Y. \blacksquare$$

Proposition 3. *If* $\forall Y,\ f\circ f^{-1}(Y) \leqq Y$, *then* a) $f^{-1}\circ f\circ f^{-1} = f^{-1}$,

b) $[f^{-1}$ *is injective*$] \Rightarrow [f$ *is surjective*$]$.

Proof. a) $\forall Y$, by Definition 1, $[f(f^{-1}(Y)) = Y_1] \Rightarrow [f^{-1}(Y) \leqq f^{-1}(Y_1)]$ while, by the hypothesis, $Y_1 \leqq Y$ and, by Definition 1, $[Y_1 \leqq Y] \Rightarrow [f^{-1}(Y_1) \leqq f^{-1}(Y)]$ so that $f^{-1}\circ f\circ f^{-1}(Y) = f^{-1}(Y)$.

b) Keeping the notation from a), $f^{-1}(Y) = f^{-1}(Y_1)$ and the hypothesis that f^{-1} is injective implies that $Y = Y_1$, hence f is surjective. $\blacksquare$

Proposition 4. *Hypotheses*: $\forall Y,\ f\circ f^{-1}(Y) \leqq Y$; f *is isotone.*

Conclusions: a) $f^{-1}(\bigwedge) = \bigwedge f^{-1}$, *i.e.*, f^{-1} *is a complete* $\bigwedge$*-morphism*;

b) $f\circ f^{-1}\circ f = f$;

c) $f(\bigvee) \leqq \bigvee f$.

Hence f *is a complete* $\bigvee$*-morphism.*

Proof. a) It suffices to prove $\geqq$; but

$$[X \leqq \bigwedge f^{-1}(Y_i)] \Rightarrow [\forall i, X \leqq f^{-1}(Y_i)] \Rightarrow [\forall i, f(X) \leqq f\circ f^{-1}(Y_i) \leqq Y_i] \Rightarrow$$
$$\Rightarrow [f(X) \leqq \bigwedge Y_i] \Rightarrow [X \leqq f^{-1}(\bigwedge Y_i)].$$

b) By the hypothesis, $\forall X,\ f\circ f^{-1}(f(X)) \leqq f(X)$, while, by Proposition 1, $f^{-1}\circ f(X) \geqq X$. Hence $f\circ f^{-1}\circ f(X) \geqq f(X)$ which proves the equality.

c) $[Y \geqq \bigvee f(X_i)] \Rightarrow [\forall i, Y \geqq f(X_i)] \Rightarrow [\forall i, f^{-1}(Y) \geqq f^{-1}\circ f(X_i) \geqq X_i] \Rightarrow \Rightarrow [f^{-1}(Y) \geqq \bigvee X_i] \Rightarrow [Y \geqq f\circ f^{-1}(Y) \geqq f(\bigvee)]. \blacksquare$

Corollary. *The hypotheses of Proposition* 4 *are equivalent to the supposition that* f *is a complete* $\bigvee$*-morphism.*

Proposition 5. *Under the hypotheses of Proposition* 4,

a) $[f(X) = Y] \Rightarrow [f\circ f^{-1}(Y) = Y]$;

b) $[f$ *is surjective*$] \Rightarrow [\forall Y, f \circ f^{-1}(Y) = Y$ *and* f^{-1} *is injective*$]$;

c) $[f^{-1}$ *is surjective*$] \Rightarrow [\forall X, f^{-1} \circ f(X) = X]$;

d) $[f$ *is injective*$] \Rightarrow [\forall X, f^{-1} \circ f(X) = X]$.

Proof. a) $[f(X) = Y] \Rightarrow [X \leqq f^{-1}(Y)]$; hence $Y = f(X) \leqq f \circ f^{-1}(Y) \leqq Y$.

b) $[f^{-1}(Y_1) = f^{-1}(Y_2)] \Rightarrow [f \circ f^{-1}(Y_1) = f \circ f^{-1}(Y_2)] \Rightarrow [Y_1 = Y_2]$.

c) $[X = f^{-1}(Y)] \Rightarrow [X \leqq f^{-1} \circ f(X) = f^{-1}(f \circ f^{-1}(Y)) \leqq f^{-1}(Y) = X]$.

d) $[f(X) = Y] \Rightarrow [f(f^{-1} \circ f(X)) = f \circ f^{-1}(Y) = Y = f(X)]$; the injectivity of f implies then the assertion.∎

Proposition 6. *If f is a complete $\bigvee$-morphism, then*

$$\forall Y \in f(P), \quad f^{-1}(Y) = \bigvee\{X \mid f(X) = Y\}.$$

Proof. Let $Y \in f(P)$ and set $E' = \{X \mid f(X) < Y\}$, $E'' = \{X \mid f(X) = Y\}$. Then obviously $E'' \neq \emptyset$, while, if $E' = \emptyset$, the assertion also is obvious. Suppose $E' \neq \emptyset$ and denote by X_i the elements of E' and by X_j those of E''; set $X' = \bigvee X_i$ and $X'' = \bigvee X_j$. Then $f(X' \vee X'') = f(\bigvee X_i) \vee f(\bigvee X_j) = (\bigvee f(X_i)) \vee (\bigvee f(X_j)) = Y$; hence $X' \vee X''$ is some X_j; therefore, $\bigvee X_j = X'' \leqq X' \vee X'' \leqq \bigvee X_j = X''$; hence $X' \vee X'' = X''$ and $X'' = f^{-1}(Y)$.∎

The following combined corollary of the preceding propositions seems to be useful.

Theorem 1. *Hypothesis*: *f is a complete $\bigvee$-morphism.*

Conclusions: a) *f^{-1} is a complete $\bigwedge$-morphism; hence, in particular, f^{-1} is isotone and $\bigvee f^{-1} \leqq f^{-1}(\bigvee)$;*

b) $\forall X, f^{-1} \circ f(X) \geqq X$ *and*, $\forall Y, f \circ f^{-1}(Y) \leqq Y$;

c) $f^{-1} \circ f \circ f^{-1} = f^{-1}$ *and* $f \circ f^{-1} \circ f = f$; *hence also* $(f \circ f^{-1})^2 = f \circ f^{-1}$ *and* $(f^{-1} \circ f)^2 = f^{-1} \circ f$;

d) $\forall Y \in f(P), f^{-1}(Y) = \bigvee\{X \mid f(X) = Y\}$;

e) *f is surjective iff f^{-1} is injective iff*, $\forall Y, f \circ f^{-1}(Y) = Y$;

f) *f is injective iff f^{-1} is surjective iff*, $\forall X, f^{-1} \circ f(X) = X$;

g) $f^{-1}(Y_u) = X_u$;

i) $f(X_0) = Y_0$.

Corollary. *$f^{-1} \circ f$ is a Kuratowski's closure operator.*

Theorem 2 (Characterization of f^{-1}). *Hypotheses*: *f is isotone and g is a single-valued isotone function $Q \to P$, such that*: $\forall X, g \circ f(X) \geqq X$ *and* $\forall Y, f \circ g(Y) \leqq Y$.

Conclusions: a) $g = f^{-1}$;

b) *f is a complete $\bigvee$-morphism.*

Proof. a) $\forall Y$, set $E_Y = \{X \mid f(X) \leqq Y\}$; by our permanent hypothesis, $E_Y \neq \emptyset$. $[X \leqq g(Y)] \Rightarrow [f(X) \leqq f \circ g(Y) \leqq Y] \Rightarrow [X \in E_Y]$; hence $\sup E_Y \geqq g(Y)$. $[f(X) \leqq \leqq Y] \Rightarrow [X \leqq g \circ f(X) \leqq g(Y)] \Rightarrow [X \leqq g(Y)]$; hence $\sup E_Y \leqq g(Y)$. Therefore, $g(Y) = \sup E_Y$; but $\sup E_Y = f^{-1}(Y)$.

b) f isotone and $\forall Y$, $f \circ f^{-1}(Y) \leqq Y$ was the hypothesis of Proposition 4; its conclusion c) is the above b).∎

Proposition 7. *When f is a complete surjective $\bigvee$-morphism, then $(f^{-1})^{-1} = f$ on $f^{-1}(Q)$; and if, in addition, $f^{-1}(Y_0) = X_0$, then $(f^{-1})^{-1} \leqq f$ on $P - f^{-1}(Q)$, with $<$ being actually possible.*

Proof. Set $g = f^{-1}$. Then $\forall Y' \in Q$, $g^{-1}(f^{-1}(Y')) = \bigvee\{Y \mid g(Y) \leqq f^{-1}(Y')\}$; $[f^{-1}(Y) \leqq f^{-1}(Y')] \Rightarrow [f \circ f^{-1}(Y) \leqq f \circ f^{-1}(Y'))] \Rightarrow [Y \leqq Y']$; and since $g(Y') = = f^{-1}(Y')$, $g^{-1}(f^{-1}(Y')) = Y'$; so much for $(f^{-1})^{-1}$ on $f^{-1}(Q)$. Now, when $f^{-1}(Y_0) = = X_0$, $\bigvee\{Y \mid g(Y) \leqq X\}$ is $\neq \emptyset$ for all X, hence $(f^{-1})^{-1}$ is defined on P; then for all X, $g^{-1}(X) = \bigvee\{Y_i \mid f^{-1}(Y_i) \leqq X \leqq f^{-1}(f(X))\} \leqq f(X)$. That the case $<$ is actually possible is shown by the following example: $P = \{X_0 < X_1 < X_u\}$, $Q = \{Y_0 < Y_u\}$ and $f(X_0) = Y_0$, $f(X_1) = f(X_u) = Y_u$; then $f^{-1}(Y_0) = X_0$, and $(f^{-1})^{-1}(X_1) = = \bigvee\{Y_i \mid f^{-1}(Y_i) \leqq X_1\} = Y_0$.∎

Note 1. Proposition 7 is the only result where the hypothesis $f^{-1}(Y_0) = X_0$ is explicitly made.

Note 2. The hypothesis of Theorem 1 is not sufficient for $f^{-1}(\bigvee) \leqq \bigvee f^{-1}$.

Example. Let $P = \{X_0 < X_1 < X_u\}$, $Q = \{Y_0 < Y_i < Y_u \ (i = 1, 2);\ Y_0 = = Y_1 \wedge Y_2, Y_1 \vee Y_2 = Y_u\}$ and $f(X_0) = Y_0$, $f(X_1) = Y_1$, $f(X_u) = Y_u$. Then $f^{-1}(Y_0) = = f^{-1}(Y_2) = X_0$, $f^{-1}(Y_1) = X_1$ and $f^{-1}(Y_u) = X_u$. Hence $f^{-1}(Y_1 \vee Y_2) = f^{-1}(Y_u) = = X_u > X_1 = X_1 \vee X_0 = f^{-1}(Y_1) \vee f^{-1}(Y_0)$.

In order to assure that $f^{-1}(\bigvee) \leqq \bigvee f^{-1}$, we shall have to introduce a new notion, that of the "f-finer covering".

Definition 2. a) Given (X, Y) with $f(X) \leqq Y$ and coverings (X_i), (Y_j) of X, Y, respectively (i.e., $\bigvee X_i \geqq X$, $\bigvee Y_j \geqq Y$), (X_i) will be called "*f-finer*" than (Y_j) iff $\forall i$, $\exists j$ such that $f(X_i) \leqq Y_j$ (for $Q = P$ and for the identical mapping $f : P \to P$, "f-finer" reduces to the classical notion of "finer").

b) (P, Q) will be said to have the property of "*f-fineness*" iff $\forall (X, Y)$ with $f(X) \leqq Y$ and $\forall$ covering (Y_j) of Y, $\exists$ a covering (X_i) of X f-finer than (Y_j).

Proposition 8. *When f is a complete $\bigvee$-morphism and (Y_j) is a covering of Y, then $[\forall X$ with $f(X) \leqq Y$, $\exists$ a covering (X_i) of X f-finer than $(Y_j)] \Rightarrow [f^{-1}(\bigvee Y_j) = = \bigvee f^{-1}(Y_j)]$.*

Proof. It suffices to prove $f^{-1}(\bigvee) \leqq \bigvee f^{-1}$. $[X \leqq f^{-1}(\bigvee Y_j)] \Rightarrow [f(X) \leqq \leqq f \circ f^{-1}(\bigvee Y_j) \leqq \bigvee Y_j]$. If (X_i) is a covering of X f-finer than (Y_j), then $\forall i$, $\exists j_i$ with

$f(X_i) \leqq Y_{j_i}$. $[\forall i, f(X_i) \leqq Y_{j_i}] \Rightarrow [\forall i, X_i \leqq f^{-1} \circ f(X_i) \leqq f^{-1}(Y_{j_i})] \Rightarrow [X \leqq \bigvee X_i \leqq$ $\leqq \bigvee f^{-1}(Y_{j_i}) \leqq \bigvee f^{-1}(Y_j)]$.∎

Theorem 3. *When f is a complete $\bigvee$-morphism, then (P, Q) has the property of f-fineness iff $f^{-1}(\bigvee) \Rightarrow \bigvee f^{-1}$, i.e., iff f^{-1} is a complete lattice morphism.*

Proof. 1) Suppose (P, Q) has the property of f-fineness and apply Proposition 8 with $Y = \bigvee Y_j$. 2) Suppose $f^{-1}(\bigvee) \leqq \bigvee f^{-1}$. Then, if $f(X) \leqq Y$ and $\bigvee Y_j \geqq Y$, $X \leqq$ $\leqq f^{-1}(Y) \leqq f^{-1}(\bigvee Y_j) \leqq \bigvee f^{-1}(Y_j)$, hence $(f^{-1}(Y_j))$ is a covering of X with $f \circ$ $\circ f^{-1}(Y_j) \leqq Y_j$.∎

Remark 1. Consequently, in the special case of point-mappings $f : P(E) \to$ $\to P(H)$, not points themselves but the f-fineness of $(P(E), P(H))$ implied by them influenced the above generalized interconnection between f and f^{-1}. Points contributed rather to the properties of the extension $f : P(E) \to P(H)$ of $\sigma : E \to H$; so, for instance, the property of f, in that special case, to be injective iff $f(\bigwedge) = \bigwedge f$, is not necessarily shared by a complete $\bigvee$-morphism $f : P \to Q$, even when (P, Q) has the property of f-fineness. Both "if" and "only if" assertions fail, as it is shown by the following examples: 1) $P = \{X_0 < X_u\}$, $Q = \{Y_0 = Y_u\}$ and $f(X_0) = f(X_u) =$ $= Y_0$; then $f(\bigwedge) = \bigwedge f$, but f is not irjective. 2) $P = \{X_0 < X_i < X_u \ (i = 1, 2);$ $X_0 = X_1 \wedge X_2, X_1 \vee X_2 = X_u\}$, $Q = \{Y_0 < Y' < Y_i < Y_u \ (i = 1, 2); Y' = Y_1 \wedge Y_2,$ $Y_1 \vee Y_2 = Y_u\}$ and $f(X_0) = Y_0$, $f(X_i) = Y_i$, $f(X_u) = Y_u$; then f is injective, but $f(X_1 \wedge X_2) = f(X_0) = Y_0 < Y' = Y_1 \wedge Y_2 = f(X_1) \wedge f(X_2)$.

Remark 2 (a "by-product" of Remark 1). The last example shows: If P is the four lattice and if for any injective complete $\bigvee$-morphism $f : P \to Q$, $f(\bigwedge) = \bigwedge f$, then Q cannot contain any sublattice of the form

More generally, choosing a certain P and imposing conditions on all functions $f : P \to Q$ of a certain species in order to obtain, from the "outside", information for the "inside" of Q, seems to be a possibly useful "external" method for studying Q.

We close now our general treatment of f and f^{-1} and, for the first time in this paper, make the supposition that P and Q are complemented. The following hypothesis suggests itself: In P, $[X \wedge X' = X_0] \Rightarrow [\forall$ complement $\complement X$ of X, $X' \leqq \complement X]$; similarly in Q. However, Huntington's theorem ([4], p. 46; [7], p. 130) asserts that P and Q are then simply Boolean algebras. So, all that remains is to search for additional properties of f and f^{-1} in that special case.

Proposition 9. *Suppose that P and Q are Boolean algebras and f a $\bigvee$-morphism $P \to Q$. Then*

a) $\forall(X_1, X_2)$, $f(X_1) - f(X_2) \leqq f(X_1 - X_2)$;

b) *if* $\{X \mid f(X) = Y_0\} = X_0$, *then* $[\forall(X_1, X_2), f(X_1) - f(X_2) = f(X_1 - X_2)] \Rightarrow$ $\Rightarrow$ [*f is injective*].

Proof. a) A direct consequence of the Boolean identity $(X_1 \wedge X_2) \vee \vee (X_1 \wedge \complement X_2) = X_1$ and the Boolean implication $[Y \vee Y' = Y''] \Rightarrow [Y'' - Y' \leqq Y]$.

b) Obvious.∎

Note 3. Conjecture: In Proposition 9, b), $\Leftarrow$ holds as well.

Proposition 10. *Suppose that P and Q are complete Boolean algebras and f a complete $\bigvee$-morphism $P \to Q$. Then, if (P, Q) has the property of f-fineness, f^{-1} transforms Q to a complete subalgebra P' of P and complements in P', Q have the following properties*:

a) $\forall Y$, $f^{-1}(\complement Y) = \complement f^{-1}(Y)$;

b) *more generally*, $\forall(Y_1, Y_2)$, $f^{-1}(Y_1 - Y_2) = f^{-1}(Y_1) - f^{-1}(Y_2)$.

Proof. a) By the isotoneity of f^{-1}, $f^{-1}(Y_0)$ and $f^{-1}(Y_u)$ is respectively the minimum and the maximum element of P'. $[Y \vee \complement Y = Y_u] \Rightarrow [f^{-1}(Y) \vee f^{-1}(\complement Y) = = f^{-1}(Y_u)]$ and $[Y \wedge \complement Y = Y_0] \Rightarrow [f^{-1}(Y) \wedge f(\complement Y) = f^{-1}(Y_0)]$, which proves the assertion.

b) $f^{-1}(Y_1 - Y_2) = f^{-1}(Y_1 \wedge \complement Y_2) = f^{-1}(Y_1) \wedge \complement f^{-1}(Y_2)$∎

One could think of applying the above facts to [12].

After this short digression to Boolean algebras, we come back to our initial P and Q (general complete lattices).

2. Topological considerations for the $\bigvee$-complete morphisms between complete lattices

Notation. $P_1(Q_1)$ is a $\bigvee$-complete sublattice of $P(Q)$ with X_0, $X_u(Y_0, Y_u)$; it will be its lattice of "open" elements.

Note 4. It is well-known that parts of General Topology have been generalized for (P, P_1) under supplementary hypotheses, especially for P_1; see, for instance, [1], [4], [6], [7] and, especially, [9] (among the initiators, M. H. Stone, A. Tarski and H. Wallman, in the years 1934–1938); however, even in the algebraically minded [5], P_1 is more special than here; we are concerned rather with $((P, P_1), (Q, Q_1))$ than with (P, P_1).

Definition 3. $\bigvee$-complete morphisms $f : P \to Q$ [with $f(P_1) \subseteq Q_1$ or with $f^{-1}(Q_1) \subseteq P_1$] will be called "*mappings*" ("*o-mappings*" or "*c-mappings*").

Definition 3 generalizes for arbitrary complete lattices the extension to power-sets of a single-valued function (in particular cases open or globally continuous single-valued function).

The composition of two mappings (o-mappings, c-mappings) gives again a mapping (o-mapping, c-mapping).

Note 5. In the case of the extension $f : P(E) \to P(H)$ of a multiple-valued function $\sigma : E \to H$ between topological spaces, the notion of a c-mapping is more general than that of an upper semicontinuous function in the sense of Berge ([3], p. 32, [4], p. 114–115), where $\forall X \in E, f(\{X\})$ has to be compact. According to [4] (p. 114), the notions of lower and upper semicontinuity of a multiple-valued function were introduced, in the thirties, independently by Bouligand and Kuratowski (see also: Kuratowski's "Topologie", II, 1961, p. 32, and Bouligand's "Titres et travaux scientifiques", 1961, p. 29).

For a generalization of topological facts concerning semicontinuous multiple-valued functions, independently of c-mappings, we refer the reader to Rosie Voreadou ([13], [14]).

Abbreviation. $\bigwedge$-complete distributivity = $\bigvee$-complete $\bigwedge$-distributivity.

Obviously, it holds the following

Metatheorem 1. *All results concerning globally continuous or open functions between topological spaces, which, together with their proofs, can be phrased exclusively in terms of open elements, without using distributivity (or with the use of distributivity or $\bigwedge$-complete distributivity) and with the use of only the above established results concerning the interconnection of f and f^{-1}, are a priori valid for $((P, P_1), (Q, Q_1))$ (where, moreover, P and Q are supposed, respectively, to be distributive or $\bigwedge$-completely distributive).*

Terminology (Bourbaki): Quasicompact = satisfying Heine-Borel-Lebesgue's axiom and not necessarily Hausdorff.

No special supposition on P or Q, P_1 or Q_1 is made in the following

Example. Let p and m be cardinals, m being sufficiently large; let q be an element of the Kurepa completion of the totally ordered set of cardinals $\leqq m$. Connexity and quasicompactness and, more generally, p-connexity and q-quasicompactness (see [16], [17] and [18]) are, substantially, notions for P_1 (or Q_1) (q-quasicompactness is even a notion for a q-complete semilattice). Then any surjective c-mapping $f : P \to Q$ such that (P, Q) has the property of f-fineness, preserves q-quasicompactness; if, in addition, $f^{-1}(Y_0) = X_0$, it also preserves p-connexity.

3. Adherent elements in not necessarily complemented lattices

Motivation. The introduction of "closed" elements in (P, P_1), when P is not necessarily complemented.

Additional notation. B, A and W denote respectively elements of P, P_1, Q_1; also, $C \in P$.

Definition 4. a) X will be called "*adherent*" to B, iff $\forall A$, $A \wedge X > X_0$ only if $A \wedge B > X_0$.

b) The join of all elements of P adherent to B will be called the "*adherence*" $\bar{B}$ of B.

Note 6. Even in the case of ordinary topology, no explicit use of adherent elements other than points or adherences (= closures) seems to have been made in the literature; consequently this simple notion seems new even in this case.

Abbreviations. X adh B = X is adherent to B; non adh = non adherent; etc.

Proposition 11. a) $\forall B$, X_0 adh B;

b) $[X > X_0] \Rightarrow [X \text{ non adh } X_0]$; *hence* $\bar{X}_0 = X_0$;

c) $\forall X$, X adh X; *hence* $X \leqq \bar{X}$ (*extensivity*);

d) $[X \geqq X'$ *and* X adh $B] \Rightarrow [X'$ adh $B]$;

e) $[C \geqq B$ *and* X adh $B] \Rightarrow [X$ adh $C]$; *hence* $[C \geqq B] \Rightarrow [\bar{C} \geqq \bar{B}]$ (*isotoneity*); *hence* $\bar{X}_1 \vee \bar{X}_2 \leqq \overline{X_1 \vee X_2}$;

f) $\forall A$, $[A \wedge B = X_0$ *and* $X \leqq A] \Rightarrow [X$ non adh $B]$.

Proof. Obvious; one uses the fact that Definition 4, a) is equivalent to: X adh B iff $[A \wedge B = X_0] \Rightarrow [A \wedge X = X_0]$.∎

Definition 5. A single-valued function $f : P \to Q$ will be called "*surjective from below*" iff $\forall X$ and $\forall Y \leqq f(X)$, $\exists X_1 \leqq X$ with $f(X_1) = Y$; then, in particular, together with every $Y \in f(P)$, all $\leqq Y$ elements of Q belong to $f(P)$. (An analogous definition of "surjective from above" is obviously possible.)

Example. The extension of any function $E \to H$ to the power sets $P(E)$ and $P(H)$ is surjective from below.

Proposition 12. *Hypothesis*: *f is a surjective from below single-valued function $P \to Q$, such that f^{-1} is a $\bigwedge$-morphism, with $f^{-1}(Y_0) = X_0$ and $f^{-1}(Q_1) \subseteq P_1$.*

Conclusion: $\forall B$, $[X$ adh $B] \Rightarrow [f(X)$ adh $f(B)]$.

Proof. $[f(X)$ non adh $f(B)] \Rightarrow [\exists W$ with $W \wedge f(X) > Y_0$ and $W \wedge f(B) = Y_0]$. Since $Y_0 < W \wedge f(X) \leqq f(X)$, $\exists X_1 < X_0$ and $\leqq X$ such that $f(X_1) = W \wedge f(X)$;

then, $f(X_1) \leqq W$, hence $X_0 < X_1 \leqq f^{-1} \circ f(X_1) \leqq f^{-1}(W)$ and also $X_0 < X_1 \wedge \wedge f^{-1}(W)$; hence, a fortiori, $f^{-1}(W) \wedge X > X_0$. On the other hand, $[W \wedge f(B) = Y_0] \Rightarrow [f^{-1}(W) \wedge f^{-1} \circ f(B) = X_0] \Rightarrow [f^{-1}(W) \wedge B = X_0]$. Hence, X non adh B, contrary to our hypothesis.∎

An immediate corollary of Proposition 12 is

Theorem 4a. *When f is a surjective from below c-mapping with $f^{-1}(Y_0) = X_0$, then f transforms adherent elements to adherent elements.*

Definitions 3 – 5 make sense also when P_1 and Q_1 are any nonvoid subsets of P and Q respectively, not necessarily lattices; and Propositions 11 and 12 together with Theorem 4a hold as well. This and the definitions of the lower and upper semicontinuity at a point for multiple-valued functions between topological spaces given in ([4], p. 114) suggest the following combined "generalization".

Definition 6. A $\bigvee$-complete morphism $f : P \to Q$ will be called "*locally c-mapping at X*" (briefly "*c at X*"), iff $[W \wedge f(X) > Y_0] \Rightarrow [\exists A$ with $A \wedge X > X_0$ and $f(A) \leqq W]$.

Special case. Let P and Q be topological spaces, with $f : P \to Q$ a single-valued function and X a singleton $\{X^*\}$; then "*c* at X" means simply "continuous at the point X^*".

An obvious consequence of Proposition 12 and Definition 6 is

Theorem 4b. *When f is a surjective from below **c** at X mapping with $f^{-1}(Y_0) = X_0$, then* $[X \text{ adh } B] \Rightarrow [f(X) \text{ adh } f(B)]$.

Note 7. By Proposition 11, the notion of adherence given in Definition 4, b) is weaker than the classical one in that $\bar{X} \leqq \bar{\bar{X}}$ and $\bar{X}_1 \vee \bar{X}_2 \leqq \overline{X_1 \vee X_2}$ (consequences of isotoneity and extensivity). In the special case of power-sets, this operation $X \mapsto \bar{X}$ (by its definition compatible with P_1) gives a "generalized topological" [11] or "hypotopological" ([15], p. 356, [8]) space which was already considered by E. Čech and B. Pospíšil (see also the references in [15] to O. Ore).

Theorem 5. *When P is $\bigwedge$-completely distributive, then* a) *any join of elements adherent to B is also adherent to B*;

b) *$B \mapsto \bar{B}$ defines a Kuratowski's closure operator* (also, such an operator in the sense of McKinsey and Tarski [10], extended to lattices).

Proof. Since b) is an immediate consequence of Proposition 11 and a), it will suffice to prove a). Let (X_i) be a family of elements adh B. Since $\bigvee(A \wedge X_i) = A \wedge (\bigvee X_i)$, $[A \wedge (\bigvee X_i) > X_0] \Rightarrow [\bigvee(A \wedge X_i) > X_0]$; but $[\forall i,\ A \wedge X_i = X_0] \Rightarrow [\bigvee(A \wedge X_i) = X_0]$; hence, $[A \wedge (\bigvee X_i) > X_0] \Rightarrow [\exists i$ with $A \wedge X_i > X_0] \Rightarrow [A \wedge B > X_0]$.∎

Hence, it obviously holds the following

Metatheorem 2. *When P and Q are $\bigwedge$-completely distributive, all topological facts, which together with their proofs can be phrased exclusively in terms of open sets and closures (without complementation) and with the use of only the above established results concerning the interconnection of f and f^{-1}, are a priori valid for (P, P_1) and $((P, P_1), (Q, Q_1))$.*

Note 8. The fact that a complete lattice is $\bigwedge$-completely distributive iff it is Brouwerian shows that the above interdiction concerning the use of complements may, sometimes, become less strict; this is a point, where, independently of our present work, special results appear in previous literature.

References

[1] *J. Benabou:* Treillis locaux. Séminaire Dubreil/Pisot, Paris, 1957/58.

[2] *C. Berge:* Théorie générale des jeux à *n* personnes. Mémor. Sci. Math. *138*, Paris, 1957.

[3] *C. Berge:* Espaces topologiques. Fonctions multivoques. Paris, 1959 et 1965.

[4] *G. Birkhoff:* Lattice Theory. 3rd ed., Amer. Math. Soc., 1967.

[5] *J. Lévy-Bruhl:* Introduction aux structures algébriques. Paris, 1968.

[6] *C. H. Dowker and D. Papert:* On Urysohn's Lemma. General Topology and its Relations to Modern Analysis and Algebra, II (Proc. Second Prague Topological Sympos., 1966). Academia, Prague, 1967, 111—114.

[7] *R. Faure et E. Heurgon:* Structures ordonnées et algèbres de Boole. Paris, 1971.

[8] *P. B. Krikelis:* Certain relations between generalized topology and universal algebra. Proc. Internat. Sympos. on Topology and its Applications (Herceg-Novi, 1968). Savez Društava Mat. Fiz. i Astronom., Belgrade, 1969, 236—237.

[9] *L. Lésieur:* Les treillis en topologie, I et II. Séminaire A. Chatelet-Dubreil, Paris, 1953/54.

[10] *J. McKinsey and A. Tarski:* The algebra of topology. Ann. of Math. *45* (1944), 142—191.

[11] *Z. P. Mamuzić:* Introduction to General Topology. 1963.

[12] *R. Sikorski:* Boolean Algebras. 1960.

[13] *R. Voreadou:* Comm. Indiv. Congrès Intern. Math. Nice, 1970, 72.

[14] *R. Voreadou:* Sur certaines propriétés des fonctions quelconques entre espaces topologiques. C. R. Acad. Athènes, 1969.

[15] *S. P. Zervos:* Aspects modernes de la localisation des zéros des polynomes d'une variable. (Thèse.) Ann. Sci. École Norm. Sup. (*3*), *77* (4) (1960), 303—410.

[16] *S. P. Zervos:* Une généralisation du théorème de Bolzano pour la connexité. C. R. Acad. Sci. Paris Sér. A—B *260* (1965), 5979—5982.

[17] *S. P. Zervos:* Une généralisation abstraite du théorème topologique de Weierstrass pour la préservation de la quasi-compacité; une notion de dimension de quasi-compacité. C. R. Acad. Sci. Paris Sér. A—B *260* (1965), 6781—6784.

[18] *S. P. Zervos:* Une définition générale de la dimension. Séminaire Delange-Pisot-Poiton, Paris, 1965/66.