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A general approach to the theory of set-valued mappings

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A GENERAL APPROACH TO THE THEORY OF SET-VALUED MAPPINGS

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Summary. Given an arbitrary set Y, a countably additive lattice L of subsets of Y, and a metric separable space X, we consider set-valued mappings $F: Y \to \mathscr{C}(X)$ (where $\mathscr{C}(X)$ is the space of all compact subsets of X) satisfying either condition (3) or (6) below. This is a far going generalization of upper resp. lower semi-continuous mappings (case where L is the lattice of all open subsets of Y). Other important applications are obtained by substituting to L the lattice of all Borel sets of additive class α, the lattice of measurable sets, of projective sets etc.

I. Introduction

1. Definitions. Let Y be a set of arbitrary elements and L a countably additive lattice (i.e. closed under countable unions) of subsets of Y containing as members the empty set and the set Y.

Denote $(-L) = \{E : (Y - E) \in L\}$.

Denote by A_σ (resp. A_δ) the lattice generated by the family of sets A and closed under countable unions (resp. intersections).

Let Z be a topological space. A mapping $f: Y \to Z$ will be called an *L-mapping*, briefly $f \in L^0$, if

(1) $$f^{-1}(G) \in L \text{ whenever } G \text{ is open in } Z;$$

equivalently: if

(2) $$f^{-1}(K) \in (-L) \text{ whenever } K \text{ is closed in } Z$$

(compare [4], Chapter IX).

2. Examples. In the following examples Y is assumed to be a topological space.

1. Let L be the lattice G of all *open* subsets of Y. Then the mapping $f: Y \to Z$ is an L-mapping iff f is continuous.

2. Let L be the lattice of all *Borel* subsets of Y. Then f is an L-mapping iff it is B-measurable (a Baire mapping).

3. Let L be the lattice of *Borel subsets of Y of additive class* $\alpha < \Omega$ (recall that the families: of open sets, of F_σ-sets, of $G_{\delta\sigma}$-sets etc. are additive Borel classes; the families of closed sets, of G_δ-sets, of $F_{\sigma\delta}$-sets etc. are multiplicative Borel classes). Here an L-mapping means B-measurable (Baire mapping) of class α. (For an outline of a theory of set-valued B-measurable mappings, see [9].)

4. Let Y be a Polish space (= complete separable) and let L denote its *nth projective class*; recall that the 0-projective class is the class of all Borel sets, the first projective class (the class of Souslin sets or A-sets) is composed of continuous images of sets of class 0, the second projective class (the class of CA-sets) is composed of the complements to the sets of the first projective class and so on; in general, the projective class $2n + 1$ consists of continuous images of sets of class $2n$ and the $2n$-class consists of complements of sets of class $2n - 1$.

Note that the projective classes are closed under countable union and countable intersection. If Z has a countable open base, then, if $f : Y \to Z$ is an L-mapping, i.e. if $f^{-1}(G)$ is of projective class n whenever G is open, then $f^{-1}(G)$ is also of class $n + 1$.

In particular, if $n = 1$, f is B-measurable (because by a known theorem of Souslin, a set which is simultaneously A and CA is a Borel set).

5. Let L be the lattice of *measurable sets* (more generally: a σ-algebra of subsets of Y). Then an L-mapping means L-measurable mapping.

II. Compact-valued mappings

We are going to consider *set-valued* mappings (called also *multifunctions*) $F : Y \to \mathscr{C}(X)$, where Y is an arbitrary set, X a normal space with a countable open base (equivalently: a metric separable space) and $\mathscr{C}(X)$ the space of all compact subsets of X. Thus $F : Y \to \mathscr{C}(X)$ means that for each $y \in Y$, $F(y)$ is a compact subset of X (of course, if X is compact, then $\mathscr{C}(X) = 2^X$, space of all closed subsets of X, and $F(y)$ is a closed subset of X).

The space $\mathscr{C}(X)$ is endowed with the Vietoris topology, which means (compare [11]) that the collection of all sets which are either of the form

(i) $\{F : F \subset G\}$ or
(ii) $\{F : F \cap G \neq \emptyset\}$,

where F is compact and G open in X, is an *open subbase* of $\mathscr{C}(X)$. Therefore, the collection of all sets of the form

(iii) $\{F : F \subset G) \cap \{F : F \cap G_1 \neq \emptyset\} \cap \ldots \cap \{F : F \cap G_n \neq \emptyset\}$

is an open base of $\mathscr{C}(X)$. Finally, since X has a *countable base*, so does $\mathscr{C}(X)$, and hence

(iv) *every open subset of* $\mathscr{C}(X)$ *is a countable union of sets of the form* (iii).

3. Definition. Given a lattice $\boldsymbol{L}$ (like in § 1), the mapping $F : Y \to \mathscr{C}(X)$ will be called of class $\boldsymbol{L}^+$ or briefly, $F \in \boldsymbol{L}^+$, if $F^{-1}(\mathscr{C}(G)) \in \boldsymbol{L}$, i.e., if

$$\{y : F(y) \subset G\} \in \boldsymbol{L} \text{ whenever } G \text{ is open in } X\,, \tag{3}$$

equivalently, if

$$\{y : F(y) \cap K = \emptyset\} \in \boldsymbol{L} \text{ whenever } K \text{ is closed in } X\,. \tag{4}$$

Symmetrically, $F \in \boldsymbol{L}_-$, if $F^{-1}(\mathscr{C}(K)) \in (-\boldsymbol{L})$, i.e. if

$$\{y : F(y) \subset K\} \in (-\boldsymbol{L}) \text{ whenever } K \text{ is closed in } X\,, \tag{5}$$

equivalently, if

$$\{y : F(y) \cap G = \emptyset\} \in (-\boldsymbol{L}) \text{ whenever } G \text{ is open in } X\,. \tag{6}$$

Let us note that, according to (1), $F \in \boldsymbol{L}^0$ iff $F^{-1}(G) \in \boldsymbol{L}$ for each G open in 2^X.

4. Examples and Remarks. 1. Let Y be a topological space, X compact and $\boldsymbol{L}$ the lattice of all open subsets of Y. Then $F \in \boldsymbol{L}^+$ means that F is upper semi-continuous and $F \in \boldsymbol{L}_-$ means that F is lower semi-continuous.

2. Let Y be a metric space, X compact and $\boldsymbol{L}$ the additive Borel class $\alpha < \Omega$. Then $F \in \boldsymbol{L}^+$ means that the sets $F^{-1}(2^G)$ are of additive class α. Similarly $F \in \boldsymbol{L}_-$ means that the sets $F^{-1}(2^K)$ are of multiplicative class α.

(Instead of $F \in \boldsymbol{L}^+$ ($F \in \boldsymbol{L}_-$) we also say that F is of Baire class α^+ (class α_-).)

5. Elementary properties of classes $\boldsymbol{L}^+$ and $\boldsymbol{L}_-$.

Theorem 1. *If F is constant, say $F(y) = K_0$ for each $y \in Y$, then $F \in \boldsymbol{L}^0$.*

Because $F^{-1}(\boldsymbol{G}) = Y$ or $\emptyset$ according to whether $K_0 \in \boldsymbol{G}$ or $K_0 \notin \boldsymbol{G}$. In both cases $F^{-1}(\boldsymbol{G}) \in \boldsymbol{L}$.

Theorem 2. *Let $f : Y \to X$ and $F(y) = \{f(y)\}$ (i.e., f is point-valued). If $F \in \boldsymbol{L}^+ \cup \boldsymbol{L}_-$, then f is an $\boldsymbol{L}$-mapping.*

This follows easily from the formula

$$\{y : F(y) \subset A\} = \{y : f(y) \in A\} = f^{-1}(A)$$

and from (3) and (1), resp. from (5) and (2) (substituting $A = G$ or $A = K$, resp.).

Theorem 3. $\boldsymbol{L}^0 \subset \boldsymbol{L}^+ \cap \boldsymbol{L}_-$. (Here $\boldsymbol{L}$ is not assumed to be countably additive.)

This follows from (1) and (3), resp. from (2) and (4), because $\mathscr{C}(G)$ is open, and $\mathscr{C}(K)$ is closed in $\mathscr{C}(X)$.

Theorem 4. $L^+ \cap L_- \subset L^0$. *Hence* $L^0 = L^+ \cap L_-$.

Proof. Let G be an open subset of $\mathscr{C}(X)$. Let $F \in L^+ \cap L_-$. We have to show that $F^{-1}(G) \in L$. Now by (iv), $F^{-1}(G)$ is a countable union of sets of the form

$$\{y : F(y) \subset G\} \cap \{y : F(y) \cap G_1 \neq \emptyset\} \cap \ldots \cap \{y : F(y) \cap G_n \neq \emptyset\} \tag{7}$$

and by (3) and (6) each of the factors of (7) is a member of L, and therefore the set (7) belongs to L. Hence $F^{-1}(G) \in L$.

Theorem 5. $L^+ \subset ((-L)_\sigma)_-$ *and* $L_- \subset ((-L)_\sigma)^+$.

Proof. 1. Let $F \in L^+$ and let K be closed in X. We have to show that $\{y : F(y) \subset K\} \in -(-L)_\sigma$, i.e. that $\{y : F(y) \subset K\} \in L_\delta$. Put $K = G_1 \cap G_2 \cap \ldots$ where G_n is open. Then

$$\{y : F(y) \subset K\} = \bigcap_n \{y : F(y) \subset G_n\} \tag{8}$$

and the proof is completed because $\{y : F(y) \subset G_n\} \in L$.

2. Let $F \in L_-$ and let G be open in X. We have to show that $\{y : F(y) \subset G\} \in \in (-L)_\sigma$. Put $G = K_1 \cup K_2 \cup \ldots$ where K_n is closed and $K_n \subset \operatorname{Int}(K_{n+1})$. If $F(y) \subset \subset G$, then — by compactness of $F(y)$ — there is n such that $F(y) \subset K_n$. Thus

$$\{y : F(y) \subset G\} = \bigcup_n \{y : F(y) \subset K_n\} \tag{8'}$$

and the proof is completed because $\{y : F(y) \subset K_n\} \in (-L)$.

Corollary 5'. *If* $L = -L$ *(i.e. if* L *is a* σ*-algebra), then* $L^+ = L_-$.

Corollary 5''. *If* $L \subset (-L)_\sigma$, *then* $(L^+ \cup L_-) \subset ((-L)_\sigma)^0$.

This follows from Theorems 4 and 5 and the formula

$$L^+ \subset ((-L)_\sigma)^+ \quad \text{and} \quad L_- \subset ((-L)_\sigma)_-$$

which is an obvious consequence of $L \subset (-L)_\sigma$.

Remark. The assumption $L \subset (-L)_\sigma$ is satisfied if, for example, L denotes the α additive Borel class. So, in particular, it follows from Corollary 5'' that semi-continuous compact-valued mappings are of the first Baire class.

6. Operations on classes L^+ and L_-.

Theorem 1. L^+ *and* L_- *are closed under the operation of finite union.*

In other terms, if $F_j \in L^+$ for $j = 0, 1$, and $F = F_0 \cup F_1$, then $F \in L^+$. Similarly, if $F_j \in L_-$, then $F \in L_-$.

Here $F = F_0 \cup F_1$ means that $F(y) = F_0(y) \cup F_1(y)$ for each $y \in Y$. (A similar meaning has $F = F_0 \cap F_1$.) The theorem follows immediately from the formula (compare [8], p. 20(2)):

$$F^{-1}(\mathscr{C}(A)) = F_0^{-1}(\mathscr{C}(A)) \cap F_1^{-1}(\mathscr{C}(A)),$$

where $A \subset X$ is open, respectively closed.

Theorem 2. *L^+ is closed under countable intersections.*

In other terms, if $F_n \in L^+$ for $n = 1, 2, \ldots$, then $(\bigcap_n F_n) \in L^+$.

The proof, completely similar to that of Theorem 8 of [9], is based on the following two general valid formulas (compare [8], p. 179(2)). Let $F = F_0 \cap F_1$. Then

$$F^{-1}(\mathscr{C}(G)) = \bigcup_{i,j}[F_0^{-1}(\mathscr{C}(G \cup R_i)) \cap F_1^{-1}(\mathscr{C}(G \cup R_j))],$$

where $R_i \cap R_j = \emptyset$, $R_1, R_2, \ldots$ being an open base of X closed under finite unions, and

$$F^{-1}(\mathscr{C}(G)) = \bigcup_n F_n^{-1}(\mathscr{C}(G)),$$

where $F = F_0 \cap F_1 \cap \ldots$ and $F_0 \supset F_1 \supset \ldots$.

Theorem 3. *L_- is closed under the operation of the closure of a countable union.*

More precisely, if $F = \overline{F_1 \cup F_2 \cup \ldots}$ and $F_n \in L_-$ for $n = 1, 2, \ldots$, then $F \in L_-$, provided $F(y)$ is compact for each $y \in Y$.

This follows from the formula (compare [8], p. 164 (iii)):

$$F^{-1}(\mathscr{C}(A)) = \bigcap_n F_n^{-1}(\mathscr{C}(A)).$$

Theorem 4. *If $F_0 \in L_-$ and $F_1 \in L^+$, then $\overline{F_0 - F_1} \in L_-$.*

In particular, $\overline{X - F_1} \in L_-$.

This follows from the lemma which we are now going to prove (compare also [8], p. 181(2)):

Lemma. *Let $F_0 : Y \to 2^X$, $F_1 : Y \to 2^X$ and $F = \overline{F_0 - F_1}$. Let K be closed in X and let $R_1, R_2, \ldots$ be the sequence of members of an open base of X such that $K \cap \overline{R}_i = \emptyset$. Then*

(9) $$F^{-1}(2^K) = \bigcap_i \{Y - [F_1^{-1}(2^{X-R_i}) - F_0^{-1}(2^{X-R_i})]\},$$

equivalently

(9′) $$Y - F^{-1}(2^K) = \bigcup_i \{F_1^{-1}(2^{X-R_i}) - F_0^{-1}(2^{X-R_i})\}$$

or

(9″) $[F_0(y) - F_1(y) \not\subset K] \equiv \exists i : [F_1(y) \cap \bar{R}_i = \emptyset$ *and* $F_0(y) \cap R_i \neq \emptyset]$.

Proof. 1. Let $F_0(y) - F_1(y) \not\subset K$, i.e., $F_0(y) \not\subset K \cup F_1(y)$. So let $x \in F_0(y)$ and $x \notin K \cup F_1(y)$. By the regularity of X, there exists a member of the base, we may call it R_i, such that

$$x \in R_i \quad \text{and} \quad \bar{R}_i \cap (K \cup F_1(y)) = \emptyset .$$

It follows that $F_1(y) \cap \bar{R}_i = \emptyset$ and $F_0(y) \cap R_i \neq \emptyset$, because $x \in F_0(y) \cap R_i$. Thus y satisfies the right member of (9″).

2. On the other hand, if y satisfies the right member of (9″) and $K \cap \bar{R}_i = \emptyset$, then $(F_0(y) - F_1(y)) \cap R_i \neq \emptyset$, because $(F_0(y) - F_1(y)) \cap R_i = (F_0(y) \cap R_i) - (F_1(y) \cap R_i) \neq \emptyset$.

Since $R_i \subset X - K$, it follows that $(F_0(y) - F_1(y)) \cap (X - K) \neq \emptyset$, which means that the left member of (9″) is fulfilled.

Remark. As seen, we don't require in our Lemma that the values of the mappings F_0 and F_1 be *compact*; they are only assumed to be *closed* in the (metric) space X.

The same remark applies, of course, to Theorem 4.

Theorem 5. L^+ *and* L_- *are closed under the operation of limit*

(10) $$F = \operatorname{Lim} F_n ,$$

the convergence being uniform.

More precisely: if $F_n \in L^+$ (resp. $F_n \in L_-$) for $n = 1, 2, \dots$, then $F \in L^+$ (resp. $F \in L_-$), provided $F(y)$ is compact for each $y \in Y$.

Theorem 5 is a direct consequence of the following lemma which we are going to prove.

Lemma. *Assume that the mappings* $F_n : Y \to \mathscr{C}(X)$, *where* $n = 1, 2, \dots$ *and where* X *is metric, satisfy the condition* (10); *in other terms, there is a sequence* $m_1 < m_2 < \dots$ *such that*

(11) $$\operatorname{dist}(F(y), F_j(y)) < 1/n \text{ for each } y \in Y \text{ and } j > m_n .$$

Then, we have, for each open G and closed K,

(12) $$\{y : F(y) \subset G) = \bigcup_n \bigcup_{j>m_n} \{y : F_j(y) \subset G_n)$$

(13) $$\{y : F(y) \subset K\} = \bigcap_n \bigcap_{j>m_n} \{y : F_j(y) \subset Q_n\} ,$$

where

$$G_n = \{x : \varrho(x, X - G) > 1/n\} \quad \textit{and} \quad Q_n = \{x : \varrho(x, K) \leqq 1/n\},$$

and thus

$$(14) \qquad G = G_1 \cup G_2 \cup \ldots, \quad G_1 \subset G_2 \subset \ldots$$

and

$$(15) \qquad K = Q_1 \cap Q_2 \cap \ldots, \quad Q_1 \supset Q_2 \supset \ldots .$$

Proof of (12). 1. Let $F(y) \subset G$. Since $F(y)$ is compact, there is by (14) an n such that $F(y) \subset G_n$. We have to show that there is $j > m_n$ such that $F_j(y) \subset G_n$. Suppose that the contrary is true, i.e., that $F_j(y) - G_n \neq \emptyset$ for each $j > m_n$; but then $\operatorname{Lim}_{j\to\infty} F_j(y) - G_n \neq \emptyset$, i.e. $F(y) \not\subset G_n$, which is a contradiction.

2. Let $F(y) - G \neq \emptyset$. Let $p \in F(y) - G$. By (11) there is, for each n and $j > m_n$, a point $p_j \in F_j(y)$ such that $|p_j - p| \leqq 1/n$, hence $\varrho(p_j, X - G) \leqq 1/n$, i.e., $p_j \notin G_n$. Thus $F_j(y) \not\subset G_n$.

Proof of (13). 1. Let $F(y) \subset K$. Suppose that contrary to (13), there are n and $j > m_n$ such that $F_j(y) \not\subset Q_n$. Let $q_j \in F_j(y) - Q_n$, i.e., $\varrho(q_j, K) > 1/n$ and consequently $\varrho(q_j, F(y)) > 1/n$. Therefore dist $(F_j(y), F(y)) > 1/n$, contrary to (11).

2. Let $F(y) - K \neq \emptyset$. Let $p \in F(y) - K$. Therefore there is n such that $\varrho(p, K) > 1/n$, i.e. $p \notin Q_n$. Suppose that for each $j > m_n$, we have $F_j(y) \subset Q_n$. Then $\operatorname{Lim}_{j\to\infty} F_j(y) \subset Q_n$, i.e., $F(y) \subset Q_n$, which is a contradiction.

Remark. A similar formula to (13) is known for point-valued mappings (see [4], p. 268, and [8], p. 386).

Theorem 6. *Let X_n be compact and let $X = X_0 \times X_1 \times \ldots$. If each mapping $F_n : Y \to 2^{X_n}$ belongs to the class L^+ (respectively to the class L_-) for $n = 0, 1, \ldots$, then so does their Cartesian product $F = F_0 \times F_1 \times \ldots$.*

The proof is completely analogous to the proof of Theorem 12 of [9].

7. Selection problems. In this section we assume that L is a σ-algebra and that X is a Polish (complete separable) space (2^X is endowed with the Vietoris topology).

According to Theorem and Corollary 1 of [10], the following Selection Theorem is true.

Theorem. *If $F : Y \to 2^X$ is either L^+ or L_-, there exists a selector $f : Y \to X$ (i.e., $f(y) \in F(y)$) which is an L-mapping.*

Corollary. *For each L-measurable mapping* $F : Y \to 2^X$ *there exists an L-measurable selector* $f : Y \to X$.

For further applications (also to the optimal control theory) see e.g. [5], [6], [7], and [12].

III. Case where Y is a topological space

8. Relations to continuous, closed, open mapping etc.

The proof of the following theorem is immediate.

Theorem 1. *Let us assume that all open subsets of* Y *are members of the lattice* $\boldsymbol{L}$. *Then each continuous mapping* $F : Y \to \mathscr{C}(X)$ *is an* $\boldsymbol{L}$*-mapping.*

More precisely: if F is upper (lower) semi-continuous, then $F \in \boldsymbol{L}^+ (F \in \boldsymbol{L}_-)$.

Corollary. *If* $Y = \mathscr{C}(X)$, *then the identity,* $F(K) = K$, *is an* $\boldsymbol{L}$*-mapping.*

Theorem 2. *Let* $f : X \to Y$ *be continuous and onto and let* $f^{-1}(y)$ *be compact for each* $y \in Y$. *Then the mapping* $f^{-1} : Y \to \mathscr{C}(X)$ *satisfies the equivalences*:

$$(f^{-1} \in \boldsymbol{L}^+) \equiv (f(K) \in (-\boldsymbol{L}) \textit{ for each closed } K \textit{ in } X),$$
$$(f^{-1} \in \boldsymbol{L}_-) \equiv (f(G) \in \boldsymbol{L} \textit{ for each open } G \textit{ in } X).$$

This follows by virtue of the general valid formula (see [8], p. 14(3)):

$$f(A) = \{y : A \cap f^{-1}(y) \neq \emptyset\}$$

in which one has to substitute for A either K or G (compare (4) and (6)).

Remark. In the case where $\boldsymbol{L}$ is the lattice of all open subsets of Y, our theorem states that f^{-1} is upper semi-continuous iff f is a closed mapping; f^{-1} is lower semi-continuous iff f is an open mapping (compare [8], p. 177).

9. The graph of the relation $x \in F(y)$. The set

$$J = \{\langle x, y\rangle : x \in F(y)\}$$

is the graph under consideration.

Theorem. *Let* M *be a lattice of subsets of* $X \times Y$, *closed under countable unions and such that*

$$(G \textit{ open in } X \textit{ and } A \in \boldsymbol{L}) \Rightarrow (G \times A) \in M\,.$$

Let $F \in \boldsymbol{L}^+$, *then* $[(X \times Y) - J] \in M$.

Proof. Let $G_1, G_2, \ldots$ be an open base of X. Then: $x \notin F(y)$ iff there is n such that $x \in G_n$ and $F(y) \cap \bar{G}_n \neq \emptyset$, i.e.,

$$(X \times Y) - J = \bigcup_n \{\langle x, y\rangle : (x \in G_n)(F(y) \cap \bar{G}_n = \emptyset)\} = \\ = \bigcup_n [G_n \times \{y : F(y) \cap \bar{G}_n = \emptyset\}].$$

Since $F \in L^+$, we have, by (4), $\{y : F(y) \cap \bar{G}_n = \emptyset\} \in L$; this completes the proof.

Corollary 1. *If F is upper semi-continuous, then J is closed in $X \times Y$.*

Here L denotes the lattice of all open subsets of Y.

Corollary 2. *If F is of Baire class α^+, then J is of Borel multiplicative class α in $X \times Y$.*

Here L denotes the lattice of Borel subsets of additive class α of Y.

Corollary 3. *If L is any projective class (in Y), then J belongs to the projective class $(-L)$ in $X \times Y$.*

Corollary 4. *If F is L-measurable, so is J.*

For the sake of simplicity we put here $X = Y =$ interval.

Remark. In the case where $F(y)$ reduces to a single point, $f(y)$, our Theorem implies some well known statements about the graph of the mapping f

$$J_0 = \{\langle x, y\rangle : x = f(y)\}$$

(see e.g. [8], p. 384 and [3], Section 3).

Let us note that the converse to Corollary 2 is not true: the set J_0 can be G_δ without f being of class 1.

IV. Final remarks

It seems useful in many cases to consider the set Φ of all mappings $F : Y \to \mathscr{C}(X)$ as a *metric space*. The distance between two members F_0 and F_1 of Φ is defined following the regular procedure (compare [8], p. 218). Namely – assuming that X is bounded – we put

$$\text{Dist}\,(F_0, F_1) = \sup \text{dist}\,[F_0(y), F_1(y)] \quad \text{where} \quad y \in Y$$

and where "dist" means the Hausdorff distance of sets.

In view of this definition, convergence in the space Φ means the uniform convergence. Thus Theorem 5 of § 6 can be restated as follows.

The sets L^+ and L_- are closed in the space Φ.

The space Φ — although not separable, in general — has a number of interesting properties. For example, if X is complete, then so is $\mathscr{C}(X)$ and consequently (see [8], p. 408) also Φ is complete.

Furthermore, the set of semi-continuous compact-valued mappings can be shown to be non-dense in the set of Baire 1st class mappings (under suitable assumptions on X and Y).

An analogous statement is true also for Baire mappings of arbitrary class α.

Let us add that "joint semi-continuity" holds under the above defined topology of the space Φ.

The proofs of these statements and of further properties of the space Φ will appear elsewhere.

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