CONTRACTION OF SOME SPACES OF HOMEOMORPHISMS

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Let $Q = \prod_{i>0} I_i$ and $s = \prod_{i>0} I^o$ where $I_i = [-1, 1]$ and $I_i^o = (-1, 1)$. Let l_2 denote separable Hilbert space. Following Anderson [1], we say a set K in X is a Z-set if K is closed and for each non-empty homotopically trivial open set U in X, $U \setminus K$ is non-empty and homotopically trivial. Some examples of Z-sets in l_2 are closed σ-compact subsets and closed sets whose projection in infinitely many directions is a point. Let $H(X)$ be the space of homeomorphisms of X onto X with the compact-open topology. Let $H_K(X) = \{h \in H(X) \mid h/K = \mathrm{id}\}$. The main result is the following:

Theorem 1. *Let $X = Q$, s, or l_2, and let K be a compact Z-set in X. Then $H_K(X)$ is contractible.*

As background to this theorem, Wong [4] showed that any homeomorphism of X is isotopic to the identity. Renz [3] observed that this process is continuous and in fact contracts $H(X)$. In a later paper [5] Wong showed that any homeomorphism of X which is the identity on a compact Z-set K, is isotopic to the identity with each level of the isotopy being the identity on K. The proof of Theorem 1 requires a non-trivial modification of Wong's technique and the use of a canonical homeomorphism extension theorem due to Chapman [2]. We also obtain the following theorem.

Theorem 2. *Let $X = s$ or l_2, and let K be a Z-set in X. If h is a homeomorphism of X such that $h/K = \mathrm{id}$, then h is isotopic to the identity via $\{H_t\}_{t\in[0,1]}$ where for each t, $H_{t/K} = \mathrm{id}$.*

Theorem 2 shows that the compactness condition is not required for K, and thus answers a question posed in Wong's paper [5]. The methods used here do not show, however, that $H_K(s)$ is contractible for K a non-compact Z-set, and this question is still open. The requirement that K be a Z-set is necessary, and some examples are given.

References

[1] *R. D. Anderson:* On topological infinite deficiency. Michigan Math. J. *14* (1967), 365—383.
[2] *T. A. Chapman:* Canonical Extensions of Homeomorphisms. (Preprint.)
[3] *P. Renz:* The contractibility of the homeomorphism group of some product spaces by Wong's method. Math. Scand. (to appear).
[4] *R. Y. T. Wong:* On homeomorphism of certain infinite-dimensional spaces. Trans. Amer. Math. Soc. *28* (1967), 140—153.
[5] *R. Y. T. Wong:* Stationary isotopes of infinite-dimensional spaces. (Preprint.)