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1. Introduction

A manifold (without boundary) is defined as a (connected) locally Euclidean space. But when is a space locally Euclidean? Which axioms or which simple geometric properties are necessary and sufficient? Substituting the axiomatic characterization for the cube I^n — cf. [4] — we obtain such an axiomatic characterization. It depends on two axiomatic properties, 2-compactness and comparability, which are claimed for some suitable subbase. We indicate in § 2, 3 how one might axiomatically characterize manifolds by already one of these, namely comparability. A comparable subbase leads naturally to the notion of the incomparability number. Iff — locally — this topological invariant equals the dimension of the space we readily obtain a manifold.

In § 4 we discuss geometric properties which characterize S^n, I^n and $\mathbf{R}^n$. The results are partially known and still far from complete as is shown by the *conjectures* mentioned at the end.

2. For an n-dimensional compact metrizable space M there exists a finite number of continuous real-valued functions which separate points, (i.e. for distinct points in M the function values are different for at least one of the functions), because M can be embedded in E^{2n+1}.

The minimal number k of such a set of separating functions equals the dimension of the Euclidean space of lowest dimension in which M can be embedded – the *m.e.d.*, minimal embedding dimension. This is also clear, because these functions induce a one-to-one continuous, hence topological, mapping of M into the product E^k. Here k = m.e.d.

Now consider a locally compact metrizable space X in which every two points have homeomorphic neighborhoods (e.g. a manifold). Moreover, assume that each point has a neighborhood for which the minimal number k of separating functions equals the dimension n of X. Hence X can be locally embedded in $E^n = E^k$. But because X is n-dimensional in every point, X contains an open n-dimensional subset of E^n – according to Brouwer's theorem on the invariance of domain –, so X contains an n-dimensional cube as neighborhood of each point. Hence X is a manifold. So it is easy to prove the following

Manifold — lemma. *A finite dimensional X is a manifold (with or without boundary), if and only if the following conditions hold*

1. *X is locally compact metrizable,*
2. *every two points have homeomorphic neighborhoods,*
3. *"the local minimal separating number" equals the dimension of X (i.e. there exists a compact neighborhood of every point for which this minimal separating number equals the dimension).*

In the next paragraph we replace the minimal separating number by an interior invariant of the space.

3. Let X be a set and $\mathfrak{U} = \{U\}$ a family of subsets U of X. $U^{(c)}$ denotes either U or U^c, the complement $X \setminus U$ of U in X. Two elements U_1 and U_2 of $\mathfrak{U}$ are called *comparable*, if

$$U_1^{(c)} \subset U_2^{(c)}$$

is true for a suitable interpretation of $U_1^{(c)}$ and $U_2^{(c)}$. Otherwise U_1 and U_2 are called incomparable. If there exists in $\mathfrak{U}$ an incomparable set (i.e. no two are comparable) of k elements (k finite), but no incomparable set of more than k elements, we call k the incomparability of $\mathfrak{U}$

$$\operatorname{inc} \mathfrak{U} = k\,.$$

We define $\operatorname{inc} \mathfrak{U} = \infty$, if no such finite k ($k = 0, 1, 2, \ldots$) exists.

We apply this notion for a topological T_1-space X and $\mathfrak{U}$ a set of generators of X (i.e. a subbase of X). We shall always assume $\emptyset, X \notin \mathfrak{U}$. Actually, we subject $\mathfrak{U}$ to two conditions. If $\mathfrak{U}$ is an *open* subbase, we require

(i) *$\mathfrak{U}$ is a T_1-subbase,* i.e. for every $U \in \mathfrak{U}$ and every point $p \in U$, there exists an $U' \in \mathfrak{U}$ such that

$$U \cup U' = X\,, \quad p \notin U'\,.$$

(ii) *Comparability condition.* $U \cup U' = X$ and $U \cup U'' = X$ in $\mathfrak{U}$, imply $U' \subset U''$ (or conversely).

The second condition can be proved to imply the transitivity of the notion of comparability in $\mathfrak{U}$. Hence $\operatorname{inc} \mathfrak{U}$ denotes the number of incomparability classes of $\mathfrak{U}$.

An open subbase $\mathfrak{U}$ satisfying (i) and (ii) is called a *comparable T_1-subbase,* if, moreover, X is T_1.

Now we define a topological invariant, the *incomparability of X* by

$$\operatorname{inc} X = \min_{\mathfrak{U} \in \Gamma} \operatorname{inc} \mathfrak{U}$$

(hence $\operatorname{inc} X$ is finite or infinite), where Γ denotes the class of all comparable T_1-subbases $\mathfrak{U}$.

Lemma. *For a (locally) compact metrizable space X we have*

$$\operatorname{inc} X = \text{m.e.d.}\, X\,.$$

The proof if this lemma runs along the same lines as a major part of the proof in [4] or [5].

Combining the preceding two lemmas we obtain

Theorem 1. *A finite-dimensional topological space X is a manifold, iff*

1. *X is locally compact metrizable,*
2. *every two points have homeomorphic neighborhoods,*
3. $$\operatorname{inc} X = \dim X\,,$$

i.e. there exists a neighborhood of a point for which inc = dim.

4. Geometric characterizations of the n-sphere $\mathbf{S}^n$, the n-cell I^n and Euclidean n-space $\mathbf{R}^n$

Definitions. A *suspension* is defined as a double cone over a common base-space.

X is an *infinite* cone if X is homeomorphic to a space which is obtained from $Z \times [0, 1)$ – where Z is *compact* –, by identifying $Z \times \{0\}$ to one point. Observe that the one-point-compactification of X is a suspension over a base-space Z' with Z' homeomorphic to Z.

Theorem 2. *A compact connected manifold (with boundary) which is a suspension, is homeomorphic to a sphere (to a cell).*

A non-compact manifold which is an infinite cone, is homeomorphic to $\mathbf{R}^n$.

The last part of this theorem is known (Rosen), and follows e.g. from a stronger theorem of Brown [cf. 2].

The sphere-case in *this* form is not known to me, but a simple proof can be based on the generalized Schoenflies theorem [1], while the theorem immediately follows from the Doyle-Hocking characterization [3].

By the one-point compactification the $\mathbf{R}^n$-case is reduced to the sphere-case. So the only – to me at least – unknown case is the cell-case, while the unifying formulation of the several characterizations should also be observed.

The proof of the cell-case depends on a variation of the generalized Schoenflies theorem (spheres replaced by cells in a proper fashion). Because such a "Schoenflies" theorem is already in existence for the infinite-dimensional case (Wong [6]), the cell-case extends to: a compact Q-manifold (i.e. every point has a neighborhood homeomorphic to the Hilbert cube Q) which is a suspension, is homeomorphic to Q

(also: a Q-manifold which is a cone, is homeomorphic to Q, because even every compact contractible Q-manifold is homeomorphic to Q according to T. A. Chapman).

Finally, it should be remarked that Theorem 2 can be strengthened in several ways.

Open, however, seems

Question. *If a compact n-manifold with boundary is a cone, is it necessarily an n-cell?*

To obtain completely satisfactory geometric characterizations the *manifold condition should be dropped completely.*

Conjectures. If a compact metrizable space is a suspension in *every* pair of points (i.e. each pair is a pair of vertices of some suspension representation) or a cone in *every* point (i.e. each point can be considered as a vertex), it is homeomorphic to a sphere or a cell.

If a compact metrizable space is both a suspension and a cone (i.e. in every pair of points, in every point respectively) it is homeomorphic to the Hilbert cube Q.

Remark. In view of Theorem 1 one has *only* to prove the manifold condition to solve the case of the sphere and of Q. However, there still arise difficulties for S^k ($k > 2$) and I^n ($n > 3$). These difficulties are not trivial. Indeed, Bing [0] constructs a "bad space", the suspension of which becomes S^4.

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