SOME OPEN QUESTIONS IN INFINITE-DIMENSIONAL TOPOLOGY

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Since the Second Prague Symposium five years ago, there have been many worthwhile results in infinite-dimensional point set topology. Open embedding theorems for manifolds modeled on Hilbert space, l_2, or on any of many other linear spaces have been established and such manifolds have been topologically classified by homotopy type. Various useful σ-compact structures for such manifolds as well as for the Hilbert cube, Q, and for Hilbert cube manifolds have been identified and studied. General homeomorphism extension theorems have been proved. And many product and factor theorems have been established and used. As reported in Mardešić's paper in these Proceedings, Chapman has recently established the equivalence of the concept of shape of compacta with the existence of homeomorphisms of certain subsets of the Hilbert cube, thus enabling the homeomorphism theory of Q to be used in settling questions of shape. Brief summaries of known results in various directions are included in [3], in [16] and in the introduction to [5].

Three working sessions identifying and listing open problems in infinite-dimensional topology have been held in Ithaca (January, 1969), in Baton Rouge (December, 1969) and in Oberwolfach (September, 1970). Based on problem lists prepared at these sessions, an extensive list of open problems [5] has been published by (and is available from) the Mathematisch Centrum, Amsterdam.

The purpose of the present paper is to identify and discuss one general and five special areas of open questions which appear both interesting and promising for future research. Of necessity, we omit many intriguing questions, e.g. questions about bundle maps over l_2-manifolds suggested by Wong's results about bundles over polyhedra cited in his paper in these Proceedings [18]. See [5] for a much more extensive list of open problems. First we give a few definitions and some notation.

In this paper, all spaces will be separable metric. Hilbert space, l_2, is the space of all square-summable sequences of reals with the norm topology, i.e., $l_2 = \{(x_i) \mid x_i$ is real and $\sum x_i^2 < \infty\}$ with $d((x_i), (y_i)) = \sqrt{\sum (x_i - y_i)^2}$. The Hilbert cube, Q, is defined as $\prod_{j>0} I_j$ where $I_j = [-1, 1]$ and $s \subset Q$ is defined as $s = \prod_{j>0} I_j^0$ where $I_j^0 = (-1, 1)$. An E-manifold, i.e., manifold modeled on a (homogeneous) space E, is a space admitting a cover by open sets homeomorphic to open subsets of E. For a compactum M, $H(M)$ is the space of homeomorphisms of M onto itself with the metric $d(h_1, h_2) = \sup_{x \in M} d(h_1(x), h_2(x))$. The symbol "$\cong$" denotes "is homeomorphic to". A set K in a space X is a *Z-set* in X if for each non-empty homotopically trivial open set $U \subset X$, $U \setminus K$ is non-empty and homotopically trivial. A σ-Z-set is a countable union of Z-sets.

For $Y_1 \subset X_1$ and $Y_2 \subset X_2$, the pair $(X_1, Y_1) \cong (X_2, Y_2)$ if there exists a homeomorphism h of X_1 onto X_2 such that $h(Y_1) = Y_2$. If X_1 and Y_1 are E-manifolds and Y_1 is a closed subset of X_1, we call (X_1, Y_1) a manifold pair.

A *Q-factor* is a space X such that $X \times Q \cong Q$.

The two dominant "building blocks" in infinite-dimensional topology are the complete but nowhere locally compact, l_2, and the compact Q. Since it is known, [1], that $l_2 \cong s$ we may regard Q as a compactification of l_2. An "almost true" meta-conjecture states that if a separable metric space is not obviously different from l_2 (or Q) then it must be homeomorphic to l_2 (or Q). The general question is to find broader and more useful characterizations of l_2 and Q. Indeed while many useful characterizations of l_2 and Q are known (we list several below), most of the specific questions which follow can be regarded as further attempts to identify (properties of) l_2 and Q or to recognize l_2 or Q as spaces arising in contexts other than the usual ones. The following theorems are among the basic characterization and representation theorems for l_2 and Q.

$Q - 1$. Every compact convex infinite-dimensional subset of l_2 (or of any linear metric space) is homeomorphic to Q, [12] and [13].

$Q - 2$. Any countable infinite product of non-degenerate Q-factors is homeomorphic to Q (and, for example, all compact contractible polyhedra or cell-complexes are Q-factors), [15] and [16].

$Q - 3$. Any contractible compactum which is locally Q is Q, [7].

$l_2 - 1$. Every separable infinite-dimensional Banach (or Fréchet) space is homeomorphic to l_2 (or to s), [1] and [11].

$l_2 - 2$. Any product of l_2 by a Q-factor or by a Q-factor slash any σ-Z-set of it is homeomorphic to l_2.

$l_2 - 3$. Any contractible separable metric space which is locally l_2 is l_2, [10].

$l_2 - 4$. Any complement of a σ-Z-set in l_2 is homeomorphic to l_2 (and, for example, every compactum in l_2 is a Z-set in l_2), [1] and [2].

$l_2 - 5$. For any two compact polyhedra K and L having no isolated points, the space of maps of K into L is homeomorphic to l_2 provided L is contractible, [9].

Problems on Spaces of Homeomorphisms

Perhaps the most far-reaching open question in the area of infinite-dimensional topology is the following:

Question. *For M any compact M-manifold, is $H(M)$ an l_2-manifold?*

An affirmative answer would permit the application or the powerful infinite-dimensional theory to questions about homeomorphisms of finite-dimensional manifolds. Many researchers in the area consider it very probable that the answer is affirmative. The affirmative answer is known (and is easy to prove) for M one-dimensional. It is also known by Geoghegan, [9], that $H(M) \times l_2 \cong H(M)$, a property that gives a partial coordinate structure to $H(M)$ and a property that is known to be shared by all l_2-manifolds. For $n \geqq 2$, one approach to the problem of whether $H(M)$ is an l_2-manifold is to attempt (1) to show that $H(M)$ is an ANR and (2) to show that if $H(M)$ is an ANR, then $H(M)$ is an l_2-manifold. For $n = 2$, Luke and Mason [14] have shown that $H(M)$ is an ANR but their argument is delicate and depends on special properties of the plane. It does not appear to generalize easily.

Other approaches to the problem have been tried but generally have led to formulations which are no more intuitively evident and perhaps no easier than the original problem. For example, one may try to identify a space of maps or embeddings which contains the space of homeomorphisms as a suitable dense subset or is contained in such space as a suitable dense subset.

Problems on Q-manifolds

A second major problem is the topological classification of Q-manifolds, i.e., manifolds modeled on the Hilbert cube.

Question. *Are all Q-manifolds characterized by proper homotopy type?* (Here "proper homotopy type" is defined in the same way as homotopy type but with all maps required to be proper.)

It is not even known whether all compact Q-manifolds are characterized by homotopy type (which for compacta is equivalent to proper homotopy type). However, by omitting compacta of various shapes from an endslice of Q, it is easy to exhibit continuumwise many Q-manifolds which are contractible and are thus of the same homotopy type but for which no two are of the same proper homotopy type.

It is interesting to note that the intuitively easier (and necessarily locally compact) Q-manifolds are not yet classified with respect to homeomorphism type whereas the obviously nowhere locally compact l_2-manifolds are homeomorphically classified by homotopy type. Perhaps the reason for this anomaly is that the local "holes" in l_2 permit many more homeomorphisms to exist and to be exhibited thus permitting many more spaces to be observed to be homeomorphic to each other (see l_2-4 above, for example).

In the proof by Henderson [10] that l_2-manifolds are characterized by homotopy type, a vital step was the open embedding theorem, namely that any l_2-manifold may be embedded as an open subset of l_2 thus yielding a single coordinate structure for the manifold. It is immediate that for a circle C, $Q \times C$ cannot be openly embedded in Q (since the boundary of the image of $Q \times C$ cannot be empty). Therefore, given a Q-manifold, M, we cannot guarantee an open embedding in Q. An affirmative answer to the following question would guarantee a (partial) coordinate structure for M and thus provide a possible substitute for the open embedding theorem. A subset of Q which contains an open set and is the product of closed non-degenerate subintervals of the coordinate intervals of Q is called a *basic closed set* in Q. A collection is star-finite if each element intersects at most finitely many elements of the collection.

Question. *For any Q-manifold M, does there exist an embedding f of M into Q such that $f(M)$ is the union of the elements of a star-finite collection of basic closed sets of Q?*

Perhaps the best two existing theorems about Q-manifolds are due to Chapman, [7], who proved (1) for any Q-manifold, M, $M \times [0, 1)$ can be openly embedded in Q and (2) for any two Q-manifolds, M_1 and M_2, $M_1 \times [0, 1) \cong M_2 \times [0, 1)$ iff $M_1 \times [0, 1)$ and $M_2 \times [0, 1)$ are of the same homotopy type. (Here $[0, 1)$ denotes the half-open interval.) The $[0, 1)$ factor kills the "proper" distinctions made by proper homotopy type for Q-manifolds.

Problems on Manifold Pairs

There are a number of interesting questions on manifold pairs which are open even for l_2-manifolds. If in the l_2-manifold pair (M, K), K is a Z-set in M then K plays the role of an abstract boundary of M since for any $p \in K$, there is an open embedding h of $l_2 \times [0, 1)$ into M such that $p \in h(l_2 \times \{0\})$ and $K \cap h(l_2 \times [0, 1)) = h(l_2 \times \{0\})$. And, conversely, a closed abstract boundary must be a Z-set.

Question. *Let (M, K) be an l_2-manifold pair with K a Z-set in M. Under what conditions can M be embedded in l_2 such that K is the topological boundary under the embedding?*

It should be remarked that for any two l_2-manifolds M_1 and M_2, there are open embeddings h_1 and h_2 of M_1 and M_2 in l_2 such that $\mathrm{Cl}\, h_1(M_1)$, $\mathrm{Cl}\, h_2(M_2)$, $\mathrm{Cl}\, h_1(M_1) \cap \mathrm{Cl}\, h_2(M_2)$ are l_2-manifolds, $l_2 = \mathrm{Cl}\, h_1(M_1) \cup \mathrm{Cl}\, h_2(M_2)$, and $h_1(M_1)$ and $h_2(M_2)$ are disjoint. Thus weird embeddings of manifold pairs in l_2 are possible.

In the l_2-manifold pair (M, K), K is said to be of finite local deficiency n at $p \in K$ if there is an open embedding h of $l_2 \times (-1, 1)^n$ into M such that $p \in l_2 \times \{0\}^n$ and $K \cap h(l_2 \times (-1, 1)^n) = h(l_2 \times \{0\}^n)$.

Question. *Given the l_2-manifold pair (M, K) with K of local deficiency n at every point of K except possibly at the points of a subset K' of K which is a Z-set both in M and in K. Is K of local deficiency n at every point?*

For $n \neq 2$ this question is open even for K' compact or consisting of a single point. For $n = 2$, Kuiper has given a counterexample using knots where K' can be a single point (or a finite cell). The answer is known to be affirmative for the analagous questions for $n = \infty$ or for $n = 0$ (using the abstract boundary $[0, 1)$ definition for $n = 0$) and indeed by the Z-set theory, the cases for $n = \infty$ and $n = 0$ are known to be different formulations of the same question.

Problems on Q-factors

In his paper in these Proceedings, West has reviewed the status of the impressive recent research on Q-factors, i.e., spaces X such that $X \times Q \cong Q$.

Question. *Characterize the Q-factors.*

Since it is clear that every Q-factor must be a compact absolute retract, we would have a characterization of Q-factors if we could answer the following question affirmatively.

Question. *Is every compact* AR *a Q-factor*?

The results cited in West's paper give many partial results and thus characterize Q-factors for certain classes of compacta, e.g., a polyhedron or a cell-complex is a Q-factor iff it is contractible.

For the general case of compact AR's, it is not intuitively clear what answer should be expected, e.g., there are compact AR's which are not local AR's, indeed, by Borsuk, [6], there is an AR, M, and a point $p \in M$ such that no open set containing p lies in an AR which is a proper subcompactum of M. If there does exist a compact AR which is not a Q-factor, then a solution of the Q-factor characterization problem might lead to interesting new classes of compacta.

Problems on Upper Semi-Continuous Decompositions of Q

A compactum K in Q is point-like if $Q \setminus K \cong Q \setminus \{pt\}$. As discussed in Mardešić's paper in these Proceedings, Chapman, [8], has proved that for any two Z-sets K_1 and K_2 in Q, $Q \setminus K_1 \cong Q \setminus K_2$ iff K_1 and K_2 have the same shape (as defined by Borsuk). Thus a Z-set in Q is point-like iff it has the shape of a point. However, compacta in Q may be point-like even though they are not Z-sets and there exist wild arcs in Q, i.e., arcs which are not point-like. In Euclidean spaces, point-like upper semi-continuous decompositions have been extensively studied and many interesting examples and theorems are known. Frequently a condition that the elements, e.g., arcs, are "tame" replaces the more general "point-like".

It seems likely that comparable studies for Q would include the hypothesis that the elements be Z-sets of trivial shape since homeomorphic Z-sets in Q are necessarily equivalently embedded and, for example, all compacta in s or in $Q \setminus s$ are Z-sets. Indeed any point-like decomposition of s is a point-like Z-set decomposition.

Let G be an upper semi-continuous decomposition of Q into Z-sets of trivial shape with hyperspace X; equivalently, let h be a map of Q onto X such that G is the collection of point-inverses under h and the elements of G are point-like Z-sets. Let H be the collection of non-degenerate elements of G. Let H^* denote the union of the elements of H.

Question. *Under what conditions can we conclude that $X \cong Q$?*

It seems likely that there exists a "dogbone" example in Q, i.e., an upper semi-continuous decomposition of Q into points and Z-set arcs such that $h(H^*)$ is a topological Cantor set and such that $X \not\cong Q$. A possible candidate for such an example is to consider two disjoint Wong-type [17] wild Cantor sets C_1 and C_2 in Q and a continuous collection of arcs joining the points of C_1 with the points of C_2 such that each arc lies except possibly for its endpoints in s (and thus is a Z-set). Let this collection of arcs be the collection H of all non-degenerate elements of G. Then, noting some special properties of Wong's construction, it seems unlikely that the elements of H can be shrunk to points in Q, i.e., that the hyperspace of G is homeomorphic to Q.

Three special cases of the general question appear interesting.

(1) If H^* lies in a Z-set (or equivalently in an endslice of Q), can we conclude that $X \cong Q$?

(2) If H^* lies in a countable union of Z-sets (or, equivalently, in $Q \setminus s$), can we conclude that $X \cong Q$?

(3) If H^* lies in s, can we conclude that $X \cong Q$?

A substantial partial result for (1) is known whereas no significant additional results are known for (2) and (3). It should be noted that for the candidate for the dogbone example above, H^* cannot lie topologically in s since then Wong's "wild" Cantor sets would be "tame".

The result known for (1) is the following:

Theorem. *If H^* is finite-dimensional, then $X \cong Q$ (and $h(H^*)$ lies in a Z-set in X)* [4].

The finite-dimensionality is probably an unnecessary hypothesis but the known proof uses the condition strongly. Notice that if for a dogbone construction with H^* zero-dimensional and the dogbone in a Z-set, then the theorem implies that $Q \cong X$. An immediate corollary of the theorem and of Chapman's characterization of shape is the following:

Corollary. *For any compactum A and map f of A onto a compactum B such that the image under f of the union of the non-degenerate point-inverses is finite-dimensional and for each $b \in B, f^{-1}(b)$ has trivial shape, then shape A = shape B.*

A result like this corollary has reportedly recently been obtained independently by Koslowski and by Sher. The argument for the corollary involves (1) embedding A in a Z-set (or endslice) in Q, (2) employing the theorem for the U.S.C. decomposition which consists of the given U.S.C. decomposition of (the image of) A and the set of individual points not in (the image of) A, and then (3) employing Chapman's characterization since the decomposition map carries Q onto Q and $Q \setminus A$ homeomorphically onto $Q \setminus B$. If the finite dimensionality condition can be omitted from the theorem, then it also can be omitted from the corollary.

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