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1. Introduction. The following two beautiful theorems were recently proved by V. V. Filippov [3, Theorems 1.1 and 2.1].

Theorem 1.1. (Filippov). *If $f : X \to Y$ is a bi-quotient s-map, and if X has a point-countable base, so does Y.*

Theorem 1.2. (Filippov). *If $f : X \to Y$ is a quotient s-map, if X has a point-countable base, and if Y is a Hausdorff space of pointwise countable type, then f is bi-quotient.*

The purpose of this note is to briefly outline a proof for Theorem 1.1 which is somewhat shorter and simpler than Filippov's (for details, see [1]), and to indicate how Theorem 1.2 can be strengthened in two directions (for details, see [4, Theorem 9.5]).

Let us briefly explain our terminology. All maps are continuous and onto. A map $f : X \to Y$ is *bi-quotient* [1] [2] if, whenever $y \in Y$ and $\mathscr{U}$ is a cover of $f^{-1}(y)$ by open subsets of X, then $y \in (\bigcup f(\mathscr{V}))^0$ for some finite $\mathscr{V} \subset \mathscr{U}$. (We use A^0 to denote the interior of A.) A map $f : X \to Y$ is an *s-map* if $f^{-1}(y)$ has a countable base for every $y \in Y$. A space Y is of *pointwise countable type* if every $y \in Y$ is contained in a compact subset K of Y of countable character in Y (i.e., there is a countable base for the neighborhoods of K in Y).

It should be remarked that, in Theorem 1.1, the cardinal $\aleph_0$ (which appears in the definition of "s-map" and "point-countable") can be replaced by any other infinite cardinal. This was also observed by Filippov.

2. A new proof for Theorem 1.1. Let $\mathscr{B}$ be a point-countable base for X, and let $\mathscr{P} = f(\mathscr{B})$. Let $\Phi = \{\mathscr{F} \subset \mathscr{P} : \mathscr{F} \text{ finite}\}$. For each $\mathscr{F} \in \Phi$, let

$$\mathscr{M}(\mathscr{F}) = \{P \in \mathscr{P} : P \subset (\bigcup\mathscr{F})^0, P \not\subset (\bigcup\mathscr{E})^0 \text{ if } \mathscr{E} \subsetneq \mathscr{F}\},$$

$$V(\mathscr{F}) = (\bigcup(\mathscr{M}(\mathscr{F})))^0 .$$

Let $\mathscr{V} = \{V(\mathscr{F}) : \mathscr{F} \in \Phi\}$. Then $\mathscr{V}$ is the required point-countable base for Y.

The verification that $\mathscr{V}$ is a base for Y is fairly routine, but the proof that it is point-countable requires some work. For details, see [1].

3. A strengthening of Theorem 1.2. Consider the following property of a space Y.

(*) If (F_n) is a decreasing sequence of subsets of Y with a common accumulation point y, then there exist closed (in Y) subsets $A_n \subset F_n$ whose union $\bigcup_{n=1}^{\infty} A_n$ is not closed in Y.

Property (*) is a useful hypothesis in a number of theorems (see [4, section 9]). Every T_1-space of pointwise countable type has property (*), but not conversely.

According to [5], a space Y is *determined by countable subsets* if a subset A of Y is closed in Y whenever $\bar{C} \subset A$ for every countable $C \subset A$. Clearly, every sequential space has this property.

We can now state the following theorem, which is easily seen to imply Theorem 1.2. The proof is given in [4, Theorem 9.5].

Theorem 3.1. *Let $f: X \to Y$ be a quotient map. Suppose that Y is a Hausdorff space satisfying (*), that X or Y is determined by countable subsets, and that $(f^{-1}(E))^-$ is Lindelöf for every countable $E \subset Y$. Then f is bi-quotient.*

References

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