J. M. van Wouwe

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Amsterdam

The class of all GO-spaces has been extensively studied by a.o. D.J. Lutzer [5] and M.J. Faber [3], and most of the material in the first section can be found there. They characterized several properties, like metrizability, perfect normality and paracompactness in terms of the order-structure. We give a characterization of generalized ordered p- and M-spaces by means of the metrizability of certain quotient-spaces.

The first section contains preliminary definitions and results, the second section contains the main results concerning p- and M-spaces. No elaborate proofs have been included.

1. Preliminaries.

Suppose $(X,\leq)$ is a linearly ordered set.

If p,q belong to X, then by the *closed interval* [p,q] we mean the set $\{x \in X \mid p \leq x \leq q\}$, and by the *open interval* (p,q) the set $\{x \in X \mid p < x < q\}$. *Half-lines*, like $\{x \in X \mid x \geq p\}$ are denoted by $[p,\rightarrow)$ etc.

A subset C of X is called *convex* if for every pair of points p,q from C the interval [p,q] is contained in C. Every subset B of X decomposes into a collection of maximal convex subsets of X, called the *convexity-components of* B (in X).

A point p is said to be an *endpoint* of a convex set C in X, if p belongs to C and $C\setminus\{p\}$ is convex.

Whenever $(X,\leq)$ is a linearly ordered set, the order-topology on X, with the open half-lines as subbase, is denoted by $\lambda(\leq)$ or simply by λ if there is no danger of confusion.

$(X,\leq,\lambda(\leq))$ is called a *linearly ordered topological space* or LOTS.

The triple $(X,\leq,\tau)$ is called a *generalized ordered topological space* or GO-space if X is a set, $\leq$ a linear order on it, and τ a topology for X such that

(i) $\lambda(\leq) \subset \tau$

(ii) τ has a base consisting of convex sets.

Clearly every subspace of a LOTS is a GO-space. The converse is also true: if $(X,\leq,\tau)$ is a GO-space, and

$$\begin{aligned} X^* = &\{(x,n) \in X \times \mathbb{Z} \mid n > 0 \text{ if } (\leftarrow,x] \in \tau\setminus\lambda\} \cup \\ &\cup \{(x,n) \in X \times \mathbb{Z} \mid n < 0 \text{ if } [x,\rightarrow) \in \tau\setminus\lambda\} \cup \\ &\cup \{(x,n) \in X \times \mathbb{Z} \mid n = 0\}, \end{aligned}$$

then X is homeomorphic to the subspace $X \times \{0\}$ of the LOTS $(X^*, \prec, \lambda(\prec))$, where $\prec$ is the lexicographic order on X^*.

Let $X = (X,\leq,\tau)$ be a GO-space.
An ordered pair (A,B) of subsets of X, such that

(i) $X = A \cup B$

(ii) $a < b$ for all $a \in A$, $b \in B$

(iii) $A,B \in \tau$

is called a *gap* : if A has no right and B has no left endpoint

a *left pseudogap* : if A ($\neq\phi$) has no right, and B has a left endpoint

a *right pseudogap*: if A has a right, and B ($\neq\phi$) has no left endpoint.

Clearly, pseudogaps do not occur in a LOTS.
If $(X,\leq,\lambda(\leq))$ is a LOTS, and $\xi = (A,B)$ is a gap in X, then we may regard ξ as a "virtual element" added to X, satisfying $a < \xi < b$ for all $a \in A$, $b \in B$. If we add all these gaps to X, and give the resulting set the ordertopology, we obtain the *Dedekind-compactification* X^+ of X [4]. If $X = (X,\leq,\tau)$ is a GO-space, we think X embedded in X^*, and define the Dedekind-compactification X^+ of X as the closure of X in the Dedekind-compactification of X^*. Then X^+ is an ordered Hausdorff-compactification of X. It can also be obtained by "adding" to X all gaps and pseudogaps, as described above.

If X and Y are topological spaces such that $X \subset Y$, then a *pluming* for X in Y is a sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ of coverings of X by open sets in Y, such that $\bigcap_{n=1}^{\infty} St(x,\mathcal{U}_n) \subset X$ for every $x \in X$.
A completely regular space X is a p-space if it has a pluming in its Čech-Stone-compactification, or equivalently, in any of its Hausdorff-compactifications (see [1]). Hence, a GO-space X is a p-space iff it has a pluming in its Dedekind-compactification X^+.
A space X is said to be a *ωΔ-space* [2] if there exists a sequence $(\mathcal{V}_n)_{n=1}^{\infty}$ of open covers of X with property (M) below:

(M) : If there exists $x_0 \in X$ such that $x_n \in St(x_0,\mathcal{V}_n)$ $(n = 1,2,\ldots)$ then the sequence $(x_n)_{n=1}^{\infty}$ has a clusterpoint.

A space X is a *M-space* [6] if it admits a normal sequence of open covers, satisfying property (M).

2. p- and M-spaces.

In the sequel, X will always denote a GO-space $(X,\leq,\tau)$.

DEFINITION If C is a convex subset of X, and $\xi = (A,B)$ is a (pseudo)gap in X, then C *covers* ξ if $C \cap A \neq \phi$ and $C \cap B \neq \phi$.

DEFINITION *Every GO-space* X*, as a subset of* X^+*, decomposes into convexity-components which are closed in* X*. Clearly the convexity-components of* X *in* X^+ *are maximal convex subsets of* X *that do not cover any (pseudo)gap of* X*. If* D *is the collection of all these convexity-components, then the decompositionspace* X/D *is called* gX*, and the identification-map is denoted by* $g : X \to gX$*.*

PROPOSITION. *If* δ *is the identification-topology on* gX*, and* $\le$ *is the natural order on* gX*, inherited from* X*, then* $gX = (gX,\le,\delta)$ *is a GO-space. Moreover, the map* $g : X \to gX$ *is closed and order-preserving.*

The main result about p-spaces is the following

THEOREM X *is a p-space* $\Leftrightarrow$ gX *is metrizable.* □

COROLLARY 1: X *is a p-space* $\Leftrightarrow$ X^* *is a p-space.*

PROOF. gX is homeomorphic to $g(X^*)$. □

COROLLARY 2: *Suppose* X *is a GO-space such that there is a (pseudo)gap between any two points of* X*. Then* X *is a p-space if and only if it is metrizable.*

PROOF. The fact that there is a (pseudo)gap between any two points of X implies that gX is homeomorphic to X. □

COROLLARY 3: *Suppose* X *is a LOTS with a* σ*-discrete dense subset. Then* X *is a p-space.*

PROOF. If D is a σ-discrete, dense subset of X, then g[D], together with possible endpoints of gX, is a σ-discrete, dense subset of gX, containing all $y \in gX$ such that $[y,\to)$ or $(\leftarrow,y]$ is an open subset of gX. Hence gX is metrizable by [3: theorem 3.1]. □

Generalized ordered M-spaces can be characterized in a way similar to that of p-spaces. We need some definitions, and the following theorem.

THEOREM. X *is a* $w\Delta$*-space* $\Rightarrow$ X *is a* M*-space.* □

DEFINITION. Suppose $\xi = (A,B)$ is a (pseudo)gap in X. Then ξ is called *countable* if some strictly increasing sequence is cofinal in A, or some strictly decreasing sequence is coinitial in B.

PROPOSITION. X *is countably compact* $\Leftrightarrow$ X *has no countable (pseudo)gaps.* □

DEFINITION. Let X^c be the subspace of X^+ containing all elements of X, and all countable (pseudo)gaps of X. For every (pseudo)gap $\xi = (A_\xi,B_\xi)$ from $X^c\backslash X$ add the set $[\xi,\to)$ of X^c to the subspace-topology of X^c if no strictly increasing sequence is cofinal in A_ξ, and the set $(\leftarrow,\xi]$ if no strictly decreasing sequence is coinitial in B_ξ. With the resulting topology, and relative order, X^c becomes a generalized ordered countably-compactification of X in the sense of Morita [7].

DEFINITION. *Let* C *be the decomposition of* X *into maximal convex sets that do not cover countable (pseudo)gaps, i.e.* C *consists of the convexity-components of* X *as a subset of* X^c. *Then the decomposition-space* X\C *is denoted by* cX *and the identification map by* $c : X \to cX$.

PROPOSITION. *If* γ *is the identification-topology on* cX, *and* $\leq$ *is the natural order on* cX *inherited from* X, *then* $cX = (cX, \leq, \gamma)$ *is a GO-space, and the map* $c : X \to cX$ *is closed and orderpreserving.* □

THEOREM. *The following properties are equivalent:*

(i) X *is an* M-*space*

(ii) X *has a pluming in* X^c

(iii) cX *is metrizable.* □

COROLLARY. X *is a* p-*space* $\Rightarrow$ X *is an* M-*space.*

PROOF. The map $h = c \circ g^{-1} : gX \to cX$ is closed, and hence preserves stratifiability. Since stratifiability is equivalent to metrizability for GO-spaces, this implies that cX is metrizable if gX is. □

The following example shows that in the class of all GO-spaces, p-spaces and M-spaces do not coincide:

EXAMPLE. Let ω_1 be the set of all countable ordinals, with the usual order, and ω_1^* the same set with reversed order. Replace in ω_1 every non-limit ordinal by a copy of $\omega_1^* + \omega_1$, and order the resulting set X lexicographically by $\prec$.
Then the LOTS $X = (X, \prec, \lambda(\prec))$ is an M-space since X has no countable gaps, so $cX = \{0\}$ (in fact X is countably compact); but X is not a p-space since gX is homeomorphic to ω_1 and hence not metrizable. □

3. References.

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