Hans-Eberhard Porst

Embeddable spaces and duality in topological categories

In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. [361]--366.

Persistent URL: http://dml.cz/dmlcz/700608

EMBEDDABLE SPACES AND DUALITY IN TOPOLOGICAL CATEGORIES

H.-E. PORST

Bremen

It was suggested during the last years to consider cartesian closed categories contained in (or containing) Top, the category of all topological spaces and continuous maps, if one wants to investigate relations between a space X and its function algebra $C(X)$. In a joint paper of the auther and M.B. Wischnewsky [8] is shown recently - generalizing an idea of Dubuc and Porta [3] - that the concept of cartesian closed topological categories is really convenient for a general Gelfand-Naimark-duality formalism.

In this note we will concentrate moreover to spaces X which are embeddable in the spectrum of their function algebra. The results contain as special instances similar ones of Binz [1] in case of limit spaces and Fröhlicher [5] in case of Kelley spaces.

Let us start with a cartesian closed topological category $T\colon \underline{V} \to \underline{Set}$, endowed with a proper (E,M)-factorization of sources [6]. The internal Hom-functors of $\underline{V}$ are denoted by $\underline{V}(V,-)$; hence $T\underline{V}(V,V')$ is the set of $\underline{V}$-maps from V to V'. If R is a $\underline{V}$-ring (i.e. in short a $\underline{V}$-object with $\underline{V}$-maps as ring-operations) then $R\text{-Alg}(\underline{V})$ denotes the category of all R-algebras in $\underline{V}$ and all R-algebra homomorphisms in $\underline{V}$. Let us recall the following fundamental facts on this category (For the language of enriched category theory we refer to [2]:

Theorem 1: [8]

(i) The category $R\text{-Alg}(\underline{V})$ is a $\underline{V}$-complete and $\underline{V}$-cocomplete $\underline{V}$-category and the underlying functor $|\ |\colon R\text{-Alg}(\underline{V}) \to \underline{V}$ is $\underline{V}$-functor which is $\underline{V}$-monadic.

(ii) $R\text{-Alg}(\underline{V})$ is cotensored, i.e. all the representables $R\text{-Alg}(\underline{V})(-,A)\colon R\text{-Alg}(\underline{V}) \to \underline{V}^{op}$ have $\underline{V}$-right-adjoints $\overline{R\text{-Alg}(\underline{V})}(-,A)$, which are defined as follows: $\overline{R\text{-Alg}(\underline{V})}(V,A)$ is the $\underline{V}$-object $\underline{V}(V,|A|)$ endowed with the R-algebra structure induced by A.

By this theorem there exists especially the V-left adjoint

$$C_R\colon \underline{V}^{op} \longrightarrow R\text{-Alg}(\underline{V})$$

of the spectral functor

$$S_R := R\text{-Alg}(\underline{V})(-,R) \ :\ R\text{-Alg}(\underline{V}) \longrightarrow \underline{V}^{op}$$

C_R is called the function algebra functor of R since $|C_R V| = \underline{V}(V, |R|)$ by 1,(ii). Let furthermore denote

$$\varepsilon: 1_V \longrightarrow S_R C_R$$

the unit of this $\underline{V}$-adjunction and

$$\eta: 1_{R\text{-Alg}(\underline{V})} \longrightarrow C_R S_R$$

the counit.

Definition: A $\underline{V}$-object V is called M-R-embeddable, iff $\varepsilon_V: V \longrightarrow SCV$ is in M, the mono-class of the given factorization on $\underline{V}$. $\underline{M_R}$ denotes the $\underline{V}$-full subcategory of $\underline{V}$ whose objects are all M-R-embeddable spaces.

Hence the c-embedded spaces in the sense of Binz [1] are M-R-embeddable (R the reals) in the category Lim of limit spaces for any (E,M)-factorization on Lim.

Proposition 1: $\underline{M_R}$ is a $\underline{V}$-full and isomorphism closed ($\underline{V}$-) E-reflective subcategory of $\underline{V}$; especially $\underline{M_R}$ is closed under the formation of (V-) limits and (V-) M-subobjects. Moreover, if $(M \xrightarrow{f_i} M_i)$ is a source in M such that all M_i are in $\underline{M_R}$, then M is in $\underline{M_R}$.

Proof: The reflection RV of a $\underline{V}$-object V is constructed by means of the (E,M)-factorization of $\varepsilon_V: V \longrightarrow RV \longrightarrow SCV$.

Corollary: A source $(X \xrightarrow{f_i} TM_i)_{i \in I}$ with M-R-embeddable objects M_i admits a T-initial lifting in $\underline{M_R}$, if there is a source $(V \xrightarrow{g_i} M_i)_{i \in I}$ in M such that $Tg_i = f_i$ for all $i \in I$.

As in case of limit spaces [1] $\underline{M_R}$ contains "all" objects related to the functors C_R and S_R:

Proposition 2: The $\underline{V}$-objects $S_R A$, $|C_R V|$, and $\underline{V}(|C_R V'|, |C_R V|)$ are M-R-embeddable for any $V, V' \in \underline{V}$ and $A \in R\text{-Alg}(\underline{V})$.

Proof: Use the adjunctions stated in theorem 1 and the cartesian structure of $\underline{V}$.

For the following let us assume, that the factorization structure on $\underline{V}$ is compatible with the internal hom-functors, i.e. that the following implication holds for any $V \in \underline{V}$:

(∗) $m \in M \Rightarrow \underline{V}(V,m) \in M$

This condition is satisfied at least in the following important cases:

(I) The $\underline{V}$-maps of M are exactly the monomorphisms

(II) The $\underline{V}$-maps of M are exactly the extremal (or equivalently regular or T-initial) monomorphisms.

Let us denote the mono-class (epiclass) of the factorization with $M^{I}(E^{I})$ resp. $M^{II}(E^{II})$ in these cases.

The use of (*), Proposition 2, and the $\underline{V}$-natural equivalence $1_{\underline{V}} \simeq \underline{V}(1,-)$ then yields:

Proposition 3: If (*) holds for any $V \in \underline{V}$, then the following assertions are equivalent:

(i) $V \in \underline{M_R}$

(ii) $\underline{V}(V',V) \in \underline{M_R}$ for all $V' \in \underline{V}$

(iii) $C_{V',V} : \underline{V}(V',V) \longrightarrow \underline{A}(CV,CV')$ is in M for any $V' \in \underline{V}$

(iv) V is an M-subobject of $|C|CV||$

These facts are stated in case of limit spaces in [1]; in [5] the implication (i) $\Rightarrow$ (ii) is proved for Kelley spaces as is the equivalence (i) $\Leftrightarrow$ (iv). Of course Proposition 3 applies to these categories.

Corollary: [cf. 5]: $\underline{M_R}$ is cartesian closed (with respect to the cartesian structure induced by $\underline{V}$).

Remark: It should be mentioned that all we have done so far works if we would start with an (E,M)-topological category over Set [6] which is cartesian closed. In this case, if the induced factorization on $\underline{V}$ satisfies condition (*), the embeddable objects with respect to this factorization again form a cartesian closed (E,M)-topological category over Set, as is clear by the preceding facts.

We now start to describe $\underline{M_R}$ as an E-reflective hull of certain $\underline{V}$-objects. The first result is an immediate consequence of Propositions 1 and 3.

Proposition 4: $\underline{M_R}$ is the E-reflective hull of the underlying objects $|C_R V|$ of all R-function algebras.

To exhibit the relation between $\underline{M_R}$ and the E-reflective hull of $|R|$ we first state the following lemma; for this recall that $T|CV| = T\underline{V}(V,|R|)$ is the set of $\underline{V}$-maps from V to $|R|$.

Lemma: The following assertions are equivalent for any V-object:

(i) $(V \xrightarrow{h} |R|)_{h \in T|CV|}$ is a mono-source [1] in a full reflective subcategory of V that contains V and |R|.

(ii) $(V \xrightarrow{h} |R|)_{h \in T|CV|}$ is a mono-source in V

(iii) $(TV \xrightarrow{Th} T|R|)_{h \in T|CV|}$ is a mono-source in Set

(iv) CV seperates points, i.e. for any pair of different V-maps $p,q\colon 1 \to V$ there exists a V-map $h\colon V \to |R|$ such that $hp \neq hq$.

Now a straightforward calculation shows that for any M^I-R-embeddable object V CV seperates points. moreover T|CV| is a mono-source for any M^I-R-embeddable object V as may be shown as in [4]. Hence we have the following result:

Proposition 5: M_R^I is the E^I-reflective hull of |R|. Moreover the following assertions are equivalent:

(i) $V \in M_R$

(ii) CV seperates points

(iii) T|CV| is a mono-source.

In case of the other factorization structure mentioned above we obtain the following result:

Proposition 6: The epireflective hull of |R| is the full subcategory of M_R^{II} whose objects are those objects V which are T-initial with respect to T|CV| (i.e. a Set-map $TV' \to TV$ is a V-map if the composita $TV' \to TV \xrightarrow{h} |R|$ are V-maps for all $h \in T|CV|$.

Proof: Recall that an embeddable object V is in the E-reflective hull of |R| iff the canonical V-map $V \to |R|^{T|CV|}$ is a monomorphism, and that this map is monic iff T|CV| is a monic-source.

An immediate consequence of Proposition 6 is the following corollary which applies for example in case of Kelley-spaces.

Corollary: M_R^{II} is the epireflective hull of |R| iff all M^{II}-R-embeddable objects V are initial with respect to T|CV|.

Remark: For a topological space X, X is initial with respect to the source C(X,ℝ) and this source is a mono-source iff X is

1) A source $(A \xrightarrow{f_i} A_i)$ is called mono-source, provided for any pair $f,g\colon B \to A$ we have $f = g$ if all the equalities $f_i f = f_i g$ $(i \in I)$ hold.

completely regular.

We omit a concretization of these results to special categories and refer to [1],[4],[5] where a lot of corresponding results can be found, which are corollaries of the ones stated here.

Let us finally show how a generalized Gelfand-Naimark duality formalism works in this context. Similarly to the definition of $\underline{M_R}$ let us define the following $\underline{V}$-full and isomorphism-closed subcategories of $\underline{V}$ resp. R-Alg($\underline{V}$):

$\underline{\text{Fix } S}$, where $V \in \underline{V}$ is in $\underline{\text{Fix } S}$ iff ε_V is an isomorphism
$\underline{\text{Fix } G}$, where $A \in$ R-Alg($\underline{V}$) is in $\underline{\text{Fix } G}$ iff η_A is an isomorphism

<u>Theorem 2:</u> $\underline{\text{Fix } S}$ and $\underline{\text{Fix } G}$ are the largest subcategories of $\underline{V}$ resp. R-Alg($\underline{V}$), such that C_R and S_R define a duality which is visualized by the following diagramm

$$\begin{array}{ccc} \underline{\text{Fix } G} & \hookrightarrow & \text{R-Alg}(\underline{V}) \\ \Big| & & S_R \downarrow \uparrow C_R \\ \underline{\text{Fix } S}^{op} & \hookrightarrow & \underline{V}^{op} \end{array}$$

If $\underline{\text{S Alg}}$ denotes the full subcategory of $\underline{V}$ whose objects are isomorphic to spectra of R-algebras in V and $\underline{CV}$ denotes the full subcategory of $\underline{\text{R-Alg}}$ whose objects are isomorphic to R function-algebras, then thefollowing result can be proved, which applies for example to limit-spaces.

<u>Proposition 7:</u> If there is an (E,M) factorization on $\underline{V}$ such that ε_V is in E for all $V \in \underline{V}$, then the following equalities hold:

(i) $\underline{\text{Fix } S} = \underline{\text{S Alg}} = \underline{M_R}$
(ii) $\underline{\text{Fix } G} = \underline{CV}$

As an example of the concrete result which may be obtained within this framework let us state the following proposition, which is also proved in [3] by transfinite construction.

<u>Proposition 8:</u> A compactly generated $\mathbb{C}$-algebra A with identity is a $\mathbb{C}$-function algebra of some Kelley space X iff A is a (pro-

jective) limit of commutative C^*-algebras in C-Alg(k-Haus).

Using classical Gelfand duality and the definition of Kelley spaces, this proposition is a consequence of the following more general result, which is to be proved by categorical routine:

Proposition 9: If D: $\underline{I} \to$ Fix G is a diagram, then the limit of D (in R-Alg($\underline{V}$)) is contained in $\underline{CV}$.

References

[1] BINZ,E.: Continuous Convergences on C(X), LN 469, Springer, Berlin,Heidelberg,New York, 1975

[2] DUBUC,E.J., PORTA,H.: Convenient categories of topological algebras and their duality theory. J. Pure Appl. Algebra 1, 281-316 (1971)

[3] DUBUC,E.J.: Kan Extensions in Enriched Category Theory, LN 145. Springer,Berlin,Heidelberg,New York 1970

[4] FRANKE,D.: Funktionenalgebren in kartesisch abgeschlossenen Kategorien. Thesis, FU Berlin 1975

[5] FRÖHLICHER,A.: Sur l'anneau des fonctions continues d'un espace à generateur compact. C.R.A.Sc. 275 (1972)

[6] HERRLICH,H.: Topological functors. Gen. Top. Appl. 4, 125-142 (1974)

[7] HERRLICH,H.: Cartesian closed topological categories. Math. Colloq. Univ. Cape Town 9, 1-16 (1974)

[8] PORST,H.-E., WISCHNEWSKY,M.B.: Gelfand monads and duality in cartesian closed topological categories. Kategorienseminar 1 Fernuniversität Hagen (1976)

Fachsektion Mathematik
Universität Bremen
2800 BREMEN
Western Germany