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A NEW METRIZATION PROOF

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0. In the early 1950's Bing [4], Nagata [11] and Smirnov [18] solved the most important open problem in General Topology, the metrization problem, by characterizing the metrizable spaces among the regular spaces by means of the existence of certain sequences of covers. Similar characterizations among the T_0 or T_1-spaces were given later by Morita [9], Stone [17], Arhangel'skii [3] and, no doubt, others. One earlier characterization among the Hausdorff spaces by Alexandroff and Urysohn [2] should also be noted. (For a brief historical account see [13] sections 1 and 2.)

Although none of these theorems is particularly easy to prove, once one is in hand the others follow readily (see [5], [13].) Hence we are justified in thinking of them as being various forms of "the metrization theorem".

The now classical method of proof consists of showing each member of each cover to be a zero set, thereby producing a family of real valued functions, by means of which the given space is embedded as a subspace of Hilbert space. (For a discussion of another avenue of proof see Rolfsen [13].)

In 1966 another important open problem was solved by Lašnev [7], who characterized the closed images of metric spaces among the T_1-spaces. Again the characterizing condition is the existence of an appropriate sequence of covers, but the technique of proof is quite different. In this case each cover is considered as a discrete space, their product is taken, and a map is constructed from a subset of this product space onto the given space. This technique is by no means new, having been used by Ponomarev [12] to characterize the first countable T_1-spaces as the open images of metric spaces, by Alexandroff and Ponomarev [1], to give an internal characterization of the dyadic bicompacta, by Slaughter [14] to show again that N^N is homeomorphic to the irrational numbers, etc.

It is the object of this note to show that this same technique will also yield a proof of the metrization theorem. This can be done directly, but in the interest of brevity and also to show the power of Lašnev's theorem we shall derive the metrization theorem from that theorem and its proof.

In section 1 we shall recall for the convenience of the reader the requisite definitions and the statements of the theorems to be used. Section 2 will be devoted to the proof. In all undefined topological terms we shall follow Dugundji [5], except that we shall write "*locally finite*" where he uses "*nbd-finite*".

1. If $x \in X$ and $\mathscr{C}$ is a cover of X, let $\mathrm{St}(x, \mathscr{C}) = \bigcup\{C \in \mathscr{C} \mid x \in C\}$. The form of the metrization theorem most suitable for our purpose is that due to Morita [9].

The metrization theorem. *A T_0-space X is metrizable if and only if there is a sequence $\{\mathscr{C}_n\}$ of locally finite closed covers of X such that for each $x \in X$ and for each open U containing x, $\mathrm{St}(x, \mathscr{C}_n) \subseteq U$ for some n.*

For Lašnev's theorem we shall require several definitions. A family of closed subsets of a space X is said to be *conservative* if the union of each subfamily is closed. The family is said to be *hereditarily conservative* if given a closed subset of each member of a subfamily, the union of these subsets is again closed. A *local net* at a point $x \in X$ is a family $\mathscr{N}$ of subsets of X each containing x such that for each neighborhood U of x there is an $N \in \mathscr{N}$ with $x \in N \subset U$. If $\mathscr{N}$ is a local net at each point of X it is called a *net* in X. A sequence $\{\mathscr{C}_n\}$ of closed covers is said to be *almost refining* if for any $x \in X$ every family $\{C_n\}$ with $x \in C_n \in \mathscr{C}_n$ is either hereditarily conservative or forms a local net at x_0. X is a *Fréchet space* if the closure of each of its subsets may be obtained by taking the limits of sequences contained in the subset. We may now state the theorem.

Lašnev's theorem. *A T_1-space X is the closed image of a metric space if and only if it is a Fréchet space and there is an almost refining sequence of hereditarily conservative closed covers whose union is a net in X.*

Finally we require a weak form of a theorem due to Morita and Hanai [10] and Stone [16].

Theorem. *The image of a metrizable space under a perfect map is itself metrizable.*

Stone's proof of the stronger version of this theorem relies on an early metrization theorem due to Mrs. Frink [6], and does not rely on the metrization theorem, otherwise our argument would be circular.

We now proceed to the proof of the metrization theorem.

2. The necessity of the condition follows trivally from the paracompactness of metrizable spaces (Stone [15]) since the open cover $\mathscr{C}'_n$ composed of the open balls of radius $1/n$, has a locally finite closed refinement $\mathscr{C}_n$ (Michael [8]) and the sequence of covers $\{\mathscr{C}_n\}$ clearly satisfy the condition.

Conversely, suppose the requisite covers exist. We may assume that each $\mathscr{C}_{n+1}$ is a refinement of $\mathscr{C}_n$, since if not we may construct a new sequence $\mathscr{C}_1, \mathscr{C}_1 \wedge \mathscr{C}_2, \mathscr{C}_1 \wedge \mathscr{C}_2 \wedge \mathscr{C}_3, \ldots$ where $\mathscr{C}_1 \wedge \ldots \wedge \mathscr{C}_n = \{C_1 \cap \ldots \cap C_n \mid C_i \in \mathscr{C}_i,\ i = 1, \ldots, n\}$ having all the desired properties. We are given that X is a T_0-space. To apply Lašnev's theorem we must show, it to be a T_1-space.

If $x, x' \in X$ and $x \neq x'$ there is an open neighborhood U of one of them, say x, which misses the other. Choose n so that $\mathrm{St}(x, \mathscr{C}_n) \subseteq U$ and let $W = X \setminus \mathrm{St}(x, \mathscr{C}_n)$.

Since $\mathscr{C}_n$ is locally finite, W is open. Now $x' \in W$ but $x \notin W$; hence X is a T_1-space. We next show it to be a Fréchet space.

Suppose x belongs to the closure of a subset A of X. Let

$$V_n = X \setminus \bigcup\{C \in \mathscr{C}_n \mid x \notin C\}.$$

Since V_n is a neighborhood of x we may choose a point $x_n \in V_n \cap A$. Since $\mathscr{C}_n$ covers X, $x_n \in \mathrm{St}(x, \mathscr{C}_n)$ and since $\mathscr{C}_{n+1}$ refines $\mathscr{C}_n$, $\mathrm{St}(x, \mathscr{C}_{n+1}) \subseteq \mathrm{St}(x, \mathscr{C}_n)$. Now if U is any open neighborhood of x we may choose n_0 so that $\mathrm{St}(x, \mathscr{C}_{n_0}) \subseteq U$. Hence for $n \geqq n_0$, $x_n \in U$, i.e. the sequence x_n is contained in A and converges to x. Hence X is a Fréchet space.

Now each $\mathscr{C}_n$, being locally finite, is hereditarily conservative. Their union is a net in X. If $x \in C_n \in \mathscr{C}_n$ for each n, $\{C_n\}$ is a local net at x. Hence all of the conditions of Lašnev's theorem are satisfied and we may conclude that there is a closed continuous map f from a metric space M onto X.

M is, in fact, that subspace of the product of the discrete spaces $\mathscr{C}_n$ consisting of those functions ∂ whose range $\{\partial(x)\}$ is a local net at some $x \in X$. $f(\partial)$ is defined to be this x. We wish to show that in our case, f is a perfect map and then apply the Morita-Hanai-Stone theorem to conclude that X is metrizable.

So, choose $x \in X$ and for each n, let $\mathscr{K}_n = \{C \in \mathscr{C}_n \mid x \in C\}$. Each $\mathscr{K}_n$ is finite and so their product $\mathbf{K}$ is compact. If $\partial \in \mathbf{K}$, $\{\partial(n)\}$ is a local net at x and so $\partial \in M$ and $f(\partial) = x$. Thus clearly $f^{-1}(x) = \mathbf{K}$ is compact, whence f is a perfect map and X is metrizable as desired.

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