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β-like compactifications

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β-LIKE COMPACTIFICATIONS

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We are concerned with a method of constructing compactifications introduced by Shanin [4], [5] and subsequently developed by various authors. All spaces under consideration will be assumed to be Hausdorff completely regular.

Let $\mathfrak{A}$ be a multiplicative (= closed under finite intersections) class of closed subsets of a space X. Shanin considers [4] a compactification cX of X such that (a) *the closures* (in cX) *of members of* $\mathfrak{A}$ *form a base for closed sets in* cX and (b) *for every* $Z_1, \dots, Z_n \in \mathfrak{A}$ *we have* $\overline{\bigcap_i Z_i}^{cX} = \bigcap_i \overline{Z}_i^{cX}$. It can be easily seen that conditions (a) and (b) are equivalent to: *for every two closed subsets,* F_1 *and* F_2 *of* X, *the closures of* F_1 *and* F_2 *in* cX *are disjoint iff there are disjoint* $Z_1, Z_2 \in \mathfrak{A}$ *with* $F_i \subset Z_i$ *for* $i = 1, 2$. If the above condition is satisfied we shall say that $\mathfrak{A}$ is a Shanin skeleton for cX. A can be a skeleton for at most one compactification and this unique compactification will be denoted by $c_{\mathfrak{A}}X$. Necessary and sufficient conditions for $\mathfrak{A}$ to be a skeleton for some compactification as well as the actual construction of $c_{\mathfrak{A}}X$ can be found in Shanin papers (Theorem 1 in [5] and Theorem 2 in [4], respectively); these details, however, are immaterial for our discussion.

It is not known whether every compactification has a Shanin skeleton. Apparently, this problem has been first stated in an explicit way in Frink [1]. It is a particular case of a more general problem; that is (speaking in very rough terms), does there exist any method which enables one to construct all compactifications from classes of sets.[1]) Numerous partial (affirmative) results have been obtained by various authors.

[1]) In 1958 we have conjectured that such a method does not exist. (In a somewhat more precise term the problem can be stated as follows: can we define two functions Φ and Ψ such that (a) Φ is defined on the class of all completely regular spaces and for every such space X, $\Phi(X)$ is a collection of classes of subsets of X; (b) Ψ is defined on the class of all pairs $(X, \mathfrak{A})$, where X is a space and $\mathfrak{A} \in \Phi(X)$, and for every fixed X, Ψ maps $\Phi(X)$ onto the totality of all compactifications of X. We require that Φ and Ψ are defined by formulas that can be written in set-theoretic and topological symbols.) Various methods of this type have been proposed by different authors; but in general, it has been shown that if a compactification can be constructed by one of these methods, then it can be constructed by the Shanin method. A possible exception to this rule is the method proposed by Fomin. Every compactification that can be constructed by Shanin method can be constructed by the Fomin method (this result is due to J. Wasilewski), but it is not known if the converse is true.

In this note we shall deal with the following: given a compactification cX we denote by $\mathfrak{A}_{cX}$ the class of all zero-sets of bounded real functions on X that can be continuously extended over cX. $\mathfrak{A}_{cX}$ is always a skeleton for some compactification; sometimes $\mathfrak{A}_{cX}$ is a skeleton for cX, but in general we have $cX \underset{\text{ext}}{\leqq} c_{\mathfrak{A}_{cX}}X$. Note, however, that the Q-closures[2]) of X in cX and $c_{\mathfrak{A}_{cX}}X$ coincide; in other words, cX and $c_{\mathfrak{A}_{cX}}X$ may differ only outside the Q-closure of X. Note also that $\mathfrak{A}_{cX}$ is always countably multiplicative (= closed under countable intersections).

Theorem 1. *The following conditions on a compactification cX are equivalent.*

(a) *cX has the following property: if f and g can be continuously extended over cX, $g(p) \neq 0$ for every $p \in X$, and f/g is bounded, then f/g can be continuously extended over cX;*

(b) *$\mathfrak{A}_{cX}$ is a skeleton for cX;*

(c) *there exists a countably multiplicative skeleton for cX.*

Furthermore, if $\mathfrak{A}$ is an arbitrary countably multiplicative skeleton for cX, then $\mathfrak{A}_{cX} \subset \mathfrak{A}$.

Compactifications satisfying the conditions of the above have many properties analogous to βX and we shall call them β-like compactifications. For instance, we have the following: *a closed G_δ-subset of a β-like compactification cX contained in $cX \setminus X$ is of cardinality $\geqq$* exp exp $\aleph_0$. The list of such analogies can be continued and further investigation in this direction seems to be worthwhile.

We shall mention some immediate corollaries of Theorem 1. *Every compactification of X is β-like iff X is extremal.*[3]) (Note that $c_{\mathfrak{A}_{cX}}X$ is always β-like.) *βX is the only β-like compactification of X iff X is Lindelöf or X admits only one compactification.*

We shall conclude with a sufficient condition for existence of a Shanin skeleton. We shall denote by $\mathfrak{F}_{cX}$ the class of all bounded real functions that can be continuously extended over cX.

Theorem 2. *Suppose that $\mathfrak{F}_{cX}$ contains a dense (in the sense of uniform convergence) subring $\mathfrak{F}$ such that $\mathfrak{F}$ contains every bounded quotient of any two of its members. Then cX has a Shanin skeleton.*

All of the above theorems are based on the results of [2] and [3]. A complete summary of these investigations will be published in Fundamenta Mathematicae.

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[2]) The Q-closure of a subset P of a space S is the set of all points $g \in S$ such that for every continuous real function f on S, if $f(p) > 0$ for every $p \in P$, then $f(g) > 0$.

[3]) A space X is called extremal (in the sense of Fréchet) provided that every continuous real function on X is bounded. The "if" part of this theorem has been noticed by various authors.

References

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