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Infinitely Many Solitary Waves in Three Space Dimensions*

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Abstract. In this paper we give some existence results obtained by V. Benci, P. d'Avenia, D. Fortunato, A. Masiello and L. Pisani about a model of Lorentz-invariant nonlinear field equation in three space dimensions which gives rise to topological solitary waves.

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1 Introduction

A *solitary wave* is a solution of a *wave equation* whose energy is finite and travels as a localized packet; a *soliton* is a solitary wave which preserves its shape after interaction, having so a *particle-like* behavior.

The soliton solutions occur in many questions of mathematical physics (nonlinear optics, classical and quantum field theory, plasma physics), chemistry and biology (see [7,8,9,10]).

For some equations, the existence of soliton solutions is guaranteed by topological constraints.

In one space dimension, the simplest example of topological solitons is given by the sine-Gordon equation

$$\psi_{tt} - \psi_{xx} + \sin\psi = 0. \tag{1}$$

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This is an overview article.

If we look for finite-energy solutions, we put the asymptotic conditions

$$\begin{cases} \psi(-\infty, t) = 2h\pi \\ \psi(+\infty, t) = 2k\pi \end{cases} \qquad h, k \in \mathbb{Z}$$

and just the difference $h - k$ represents the topological constraint.

It is well known that the only static solutions of the sine-Gordon equation

$$u_K(x) = 4\arctan\left(e^{x+a}\right) + 2h\pi$$
$$u_{AK}(x) = 4\arctan\left(e^{-x+a'}\right) + 2h'\pi$$

give rise to one-soliton solutions.

Indeed, since the sine-Gordon equation is Lorentz-invariant, we consider the "*travelling*" solutions

$$\psi_K(x,t) = u_K\left(\frac{x - vt}{\sqrt{1 - v^2}}\right)$$
$$\psi_{AK}(x,t) = u_{AK}\left(\frac{x - vt}{\sqrt{1 - v^2}}\right)$$

with $v \in \mathbb{R}, |v| < 1$.

Moreover, there exist other solutions of (1) which represent the *superposition* of these basic solutions.

This kind of results lead Derrick to look for stable, time-independent, localized solutions of the nonlinear wave equation

$$\psi_{tt} - \Delta\psi + V'(\psi) = 0 \tag{2}$$

in three space dimensions.

In [6] he proves that the corresponding static equation

$$-\Delta u + V'(u) = 0$$

has not finite-energy stable solutions.

In the same paper he proposes several ways to avoid this difficulty.

One of them is the following.

The equation (2) is the Euler-Lagrange equation related to the action

$$\mathcal{S}_1 = \iint \mathcal{L}_1 dxdt$$

where,

$$\mathcal{L}_1 = -\frac{1}{2}\sigma - V(\psi),$$

being

$$\sigma = |\nabla\psi|^2 - |\psi_t|^2.$$

Derrick suggests to take a Lagrangian density

$$\mathcal{L}_1 = -\alpha(\sigma) - V(\psi)$$

which gives rise to Lorentz-invariant equation, but he concludes that such kind of Lagrangian density lead to a very complicated differential equation. Indeed, in the 60ties, the methods of Nonlinear Analysis were not sufficiently developed to face quasilinear equations.

In this review, we recall some existence results concerning a nonlinear wave equation which is similar to the one proposed by Derrick.

2 The model

Let us consider the internal parameter space

$$\mathcal{M} = \mathbb{R}^4 \backslash \left\{ \overline{\xi} \right\}$$

where

$$\overline{\xi} = (1, 0, 0, 0).$$

The topological solitary waves introduced in [3] are fields

$$\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathcal{M}.$$

The authors of [3] take the Lagrangian density

$$\mathcal{L}_1 = -\frac{1}{2}\left(\sigma + \frac{\varepsilon}{3}\sigma^3\right) - V(\psi)$$

where $\varepsilon > 0$ and

$$V : \mathcal{M} \to \mathbb{R}.$$

We notice that this Lagrangian density is the one related to (2) plus the *correction term*

$$-\frac{\varepsilon}{6}\sigma^3.$$

The Euler-Lagrange equation related to the action

$$\mathcal{S}_1 = \iint \mathcal{L}_1 dx dt$$

is

$$\frac{\partial}{\partial t}\left(\left(1 + \varepsilon\sigma^2\right)\psi_t\right) - \nabla\left(\left(1 + \varepsilon\sigma^2\right)\nabla\psi\right) + V'(\psi) = 0. \tag{3}$$

Since $\mathcal{M}$ has non-trivial topology ($\pi_3(\mathcal{M}) = \mathbb{Z}$), if we suppose that the fields ψ are smooth and

$$\lim_{|x|\to\infty} \psi(x,t) = 0, \tag{4}$$

we can classify them in the following way.

Definition 1. For every $t \in \mathbb{R}$, the topological charge of $\psi(\cdot, t)$ is

$$\mathrm{ch}\,(\psi(\cdot,t)) = \deg\left((P \circ \psi)(\cdot,t), K(\psi(\cdot,t)), N\right)$$

where

$$P : \xi \mapsto \left|\overline{\xi}\right| \frac{\xi - \overline{\xi}}{\left|\xi - \overline{\xi}\right|} + \overline{\xi}$$

is the projection into the unitary sphere centered in $\overline{\xi}$,

$$N = 2\overline{\xi}$$

is the north pole of this sphere and

$$K(\psi(\cdot,t)) = \left\{x \in \mathbb{R}^3 \mid |\psi(\cdot,t)(x)| > 1\right\}.$$

If (4) is uniform with respect to t, the charge does not depend on t.

The existence results of topological solitary waves can be generalized to an arbitrary number of space dimensions with a more general choice of Lagrangian density as in [1]. The three space dimensional case is necessary to interpret the fileds ψ as charged relativistic particles.

3 The static solutions

The static solutions $u = u(x)$ of (3) solve

$$-\nabla\left(\left(1 + \varepsilon |\nabla u|^4\right)\nabla u\right) + V'(u) = 0$$

or, briefly,

$$-\Delta u - \varepsilon \Delta_6 u + V'(u) = 0. \tag{5}$$

Assume that

(V1) $V \in C^2(\mathcal{M}, \mathbb{R})$;
(V2) $V(\xi) \geq V(0) = 0$ and the Hessian matrix $V''(0)$ is non-degenerate;
(V3) there exist $c, r > 0$ such that

$$|\xi| < r \Rightarrow V\left(\xi + \overline{\xi}\right) \geq c|\xi|^{-6}.$$

We can obtain solutions of (5) looking for critical points of the functional

$$E(u) = \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u|^2 + \frac{\varepsilon}{6}|\nabla u|^6 + V(u)\right] dx.$$

In order to get

$$\int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u|^2 + \frac{\varepsilon}{6}|\nabla u|^6\right] dx < +\infty,$$

we consider the Banach space

$$H = H^1\left(\mathbb{R}^3, \mathbb{R}^4\right) \cap W^{1,6}\left(\mathbb{R}^3, \mathbb{R}^4\right).$$

By Sobolev embedding theorems we can say that the elements of H are continuous functions which go to zero at infinity.

For every u in the open subset

$$\Lambda = \left\{u \in H \mid \forall x \in \mathbb{R}^3 : u(x) \neq \overline{\xi}\right\}$$

the topological charge is well defined and the condition (V3) implies that if $u \notin \Lambda$, then

$$E(u) = +\infty.$$

Although the functional E is weakly lower semicontinuous, we cannot minimize it in the connected components

$$\Lambda_K = \{u \in \Lambda \,|\, \mathrm{ch}(u) = K\}$$

since the domain $\mathbb{R}^3$ is not compact and Λ_K are not weakly closed.

The first existence result can be stated as follows (Theorem 2.2 of [3]).

Theorem 2. *If V satisfies* (V1), (V2), (V3), *then there exists a weak solution of* (5) *obtained as minimum of the functional E in*

$$\Lambda^* = \{u \in \Lambda \mid \mathrm{ch}(u) \neq 0\}.$$

This result is proved by using a *Splitting Lemma* in the spirit of concentration-compactness principle.

Moreover we have the following result.

Theorem 3. *If V satisfies* (V1), (V2), (V3) *and for every $g \in O(3)$, and $\xi = \left(\xi^0, \xi^1, \xi^2, \xi^3\right) \in \mathcal{M}$*

$$V\left(\xi^0, g \cdot \left(\xi^1, \xi^2, \xi^3\right)\right) = V\left(\xi^0, \xi^1, \xi^2, \xi^3\right),$$

then for every $N \in \mathbb{Z}$ there exists u_N non-trivial solution of (5) *such that* $\mathrm{ch}(u_N) = N$.

For $N \neq 0$, the existence of a non-trivial solution is proved in [1]. The authors use a suitable invariance of the functional E in order to avoid the lack of compactness.

For $N = 0$ the existence of a nontrivial solution is obtained in [5] using the Hopf invariant.

A function which satisfies all these conditions is

$$V(\xi) = \omega_0^2 \left(|\xi|^2 + \frac{|\xi|^4}{\left|\xi - \overline{\xi}\right|^6}\right).$$

4 Interaction with the electromagnetic field

4.1 Charge density and current density

If we want to interpret the fields ψ as charged relativistic particles, it is natural to attribute to the topological charge the meaning of electric charge.

In [2], the authors introduce the fields

$$\mathbf{J}(\psi) : \mathbb{R}^4 \to \mathbb{R}^3$$
$$\rho(\psi) : \mathbb{R}^4 \to \mathbb{R}$$

with the meaning of charge density and current density generated by ψ.

Let

$$\eta = \sum_{k=0}^{3} \eta_k(\xi)\, d\xi^0 \wedge \ldots \wedge \widehat{d\xi^k} \wedge \ldots \wedge d\xi^3$$

be the unique 3-form closed but not exact on $\mathcal{M}$, where the hatted symbols are omitted,

$$\eta_k(\xi) = \frac{1}{|\Sigma|} \frac{(-1)^k \left(\xi^k - \bar{\xi}^k\right)}{\left|\xi - \bar{\xi}\right|^4}$$

and $|\Sigma|$ is the measure of the unitary sphere in $\mathbb{R}^4$.

Let $\psi^*\eta$ denote the pullback of η by ψ.

The Hodge operator applied to $\psi^*\eta$ gives a 1-form on $\mathbb{R}^4$

$$* (\psi^*\eta)\,. \tag{6}$$

The fields $(\mathbf{J}, \rho)$ introduced in [2] are the components of (6).

Indeed we can verify that

$$\mathrm{ch}(\psi(\cdot, t)) = \int_{\mathbb{R}^3} \rho(\psi)\, dx$$

(see Appendix of [4]).

On the other hand, since η is closed, the pullback $\psi^*\eta$ is closed and then

$$d(\psi^*\eta) = 0.$$

This can be written as the continuity equation

$$\nabla \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

Remark 4. When we consider static fields $u = u(x)$, since in the expression of $J_i(\psi)$ appears the factor $\frac{\partial \psi^j}{\partial t}$, we have

$$\mathbf{J} = \mathbf{0},$$

namely, as it is natural, in this case there is not electric current.

4.2 The system solitary wave-e.m. field

Let $(\mathbf{A}, \phi)$ denote the gauge potential associated to the electromagnetic field.

Using $(\mathbf{A}, \phi)$, we can define the electric field $\mathbf{E}$ and the magnetic induction field $\mathbf{B}$ as follows

$$\mathbf{E} = -\left(\mathbf{A}_t + \nabla\phi\right) \tag{7}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{8}$$

By (7) and (8) we get immediately the first two Maxwell equations

$$\nabla \times \mathbf{E} + \mathbf{B}_t = 0$$

$$\nabla \cdot \mathbf{B} = 0.$$

In the vacuum, with a suitable choice of the physical constants, the second pair of the Maxwell equations

$$\nabla \mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{B} - \mathbf{E}_t = 4\pi \mathbf{J}$$

can be obtained as the Euler-Lagrange equations related to the action

$$\mathcal{S}_{emf} = \iint \left(\mathcal{L}_2 + \mathcal{L}_3\right) dxdt$$

where

$$\mathcal{L}_2 = \frac{1}{8\pi}\left(\left|\mathbf{E}\right|^2 - \left|\mathbf{B}\right|^2\right)$$

is the Lagrangian density of the electromagnetic field and

$$\mathcal{L}_3 = (\mathbf{J}|\mathbf{A}) - \rho\phi.$$

Hence, if we consider the system solitary wave-electromagnetic field, the total action is

$$\mathcal{S} = \mathcal{S}\left(\psi, \mathbf{A}, \phi\right) = \mathcal{S}_1\left(\psi\right) + \mathcal{S}_{emf}\left(\psi, \mathbf{A}, \phi\right).$$

The Euler-Lagrange equations in the static case are

$$-\Delta u - \varepsilon\Delta_6 u + V'\left(u\right) = G \tag{9}$$

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = 0 \tag{10}$$

$$-\Delta\phi = 4\pi\rho\left(u\right) \tag{11}$$

where G derives from the interaction term and depends on u, ϕ (and their derivatives).

We have the following results:

Theorem 5. *If V satisfies* (V1), (V2), (V3), *then there exist*

$$u \in H \text{ and } \phi \in D^{1,2}\left(\mathbb{R}^3, \mathbb{R}\right)$$

such that

– $\mathrm{ch}\,(u) \neq 0$;
– $(u, 0, \phi)$ *is solution of (9, 10, 11).*

(Theorem 1.1 of [2])

Theorem 6. *If V satisfies* (V1), (V2), (V3), *and for every $g \in O\,(3)$, and $\xi = \left(\xi^0, \xi^1, \xi^2, \xi^3\right) \in \mathcal{M}$*

$$V\left(\xi^0, g \cdot \left(\xi^1, \xi^2, \xi^3\right)\right) = V\left(\xi^0, \xi^1, \xi^2, \xi^3\right),$$

then for every $N \in \mathbb{Z}$ there exist

$$u_N \in H \text{ and } \phi_N \in D^{1,2}\left(\mathbb{R}^3, \mathbb{R}\right)$$

such that

– $\mathrm{ch}\,(u_N) = N$;
– $(u_N, 0, \phi_N)$ *is a non-trivial solution of (9, 10, 11).*

(Theorem 7 of [4])

The static solutions of (9, 10, 11) are obtained as critical points of the functional

$$f\,(u, \phi) = \int_{\mathbb{R}^3} \left(\frac{1}{2}\left|\nabla u\right|^2 + \frac{\varepsilon}{6}\left|\nabla u\right|^6 + V\,(u)\right) dx - \frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla \phi\right|^2 dx + \int_{\mathbb{R}^3} \phi \rho\,(u)\, dx$$

which is strongly indefinite.

Then the solutions are obtained using the *reduced* functional

$$J\,(u) = f\,(u, \varPhi\,[u])$$

where $\varPhi\,[u]$ is implicitly defined by

$$\frac{\partial f}{\partial \phi} = 0.$$

Remark 7. If $\bar{u}$ is the non-trivial static solution of (5) having $\mathrm{ch}(\bar{u}) = 0$ found in Theorem 2 of [5], an immediate calculation shows that $(\bar{u}, 0, 0)$ is a solution of (9, 10, 11) (Theorem 4 of [5]).
This is a further confirmation of the model's coherence: indeed an uncharged particle does not create any electromagnetic field.

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