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Bifurcation of periodic points and normal form theory in reversible diffeomorphisms.

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Abstract. We survey a number of results on the bifurcation of periodic points from a fixed point in parametrized families of reversible diffeomorphisms; such problems arise for example when stufying subharmonic branching in reversible systems. We provide a structure preserving reduction result which can be used to study 'branching phenomena' near a fixed point. We also briefly discuss how one can determine the stability of bifurcating periodic orbits using normal form theory. Here an improvement of a previous normal form result is given. As an application we give the analysis of the branching of subharmonic solutions from a primary branch of periodic solutions of a reversible system.

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1 Set up and basic reduction result

Consider a m-parameter family Φ_λ of reversible (local) diffeomorphisms on $\mathbb{R}^n$ having a fixed point at the origin, i.e. $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $(x, \lambda) \mapsto \Phi_\lambda(x)$ is such that

(H) - $\Phi(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}^m$;

- $A_\lambda := D_x\Phi_\lambda(0) \in \mathcal{L}(\mathbb{R}^n)$ is invertible for all $\lambda \in \mathbb{R}^m$;

(R) there exists a linear involution $R \in \mathcal{L}(\mathbb{R}^n)$ (i.e. $R^2 = Id$) such that

$$R \cdot \Phi_\lambda \cdot R = \Phi_\lambda^{-1}, \quad \forall \lambda \in \mathbb{R}^m. \tag{1}$$

This is an overview article.

Such reversible diffeomorphisms arise for example as stroboscopic maps for periodic non-autonomous time-reversible systems, or as Poincaré-maps for symmetric periodic solutions of autonomous reversible systems. We study then the following bifurcation problem.

(P) Given an integer $q \geq 1$ and some $\lambda_0 \in \mathbb{R}^m$, find all the solutions $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ near $(0, \lambda_0)$ of the equation

$$\Phi_\lambda^q(x) = x; \qquad \Phi_\lambda^q = \Phi_\lambda \circ \cdots \circ \Phi_\lambda \ (q\text{-times}). \tag{2}$$

Without loss of generality we can set $\lambda_0 = 0$. Let $A_0 = S_0 + N_0$ be the unique semisimple-nilpotent decomposition of A_0 (i.e. S_0 semisimple, N_0 nilpotent and $S_0 N_0 = N_0 S_0$). Setting $\mathcal{N}_0 := \log\left(I + S_0^{-1} N_0\right)$ we also see that A_0 can be written in a unique way as

$$A_0 = S_0 e^{\mathcal{N}_0}, \tag{3}$$

with $\mathcal{N}_0$ nilpotent and $S_0\mathcal{N}_0 = \mathcal{N}_0 S_0$; we call (3) the semisimple-unipotent decomposition of A_0. One can easily verify that

$$R S_0 R^{-1} = S_0^{-1} \quad \text{and} \quad R\mathcal{N}_0 = -\mathcal{N}_0 R. \tag{4}$$

Introduce then the so-called *reduced phase space* for our problem; this is a subspace of $\mathbb{R}^n$ defined by

$$U := \ker\left(S_0^q - I\right). \tag{5}$$

Note that U is the generalized eigenspace corresponding to those eigenvalues of A_0 which are q-th roots of unity. Since S_0 is semisimple we have that $\mathbb{R}^n = U \oplus \operatorname{Im}\left(S_0^q - I\right)$; also, U is invariant under R. Moreover, S_0 generates on U a $\mathbb{Z}_q$-action. It is shown in [2] that problem **(P)** reduces via an adapted Liapunov-Schmidt method to solving an appropriate *determining equation* (see equation (6) below). A particular feature of this reduction is that it does not require any restriction on the eigenvalues of A_0, i.e. higher multiplicities and nilpotencies are allowed. Also, the symmetry of the reduced diffeomorphism results in a very easy form of the bifurcation equations. In the following theorem we summarize this reduction result, referring to [2] for more details.

Theorem 1. *Assume **(H)** and fix some $q \geq 1$. Then there exist smooth mappings $x^* : U \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $\Phi_r : U \times \mathbb{R}^m \longrightarrow U$, such that the following hold (we set $\Phi_{r,\lambda} := \Phi_r(\cdot, \lambda)$):*

1. *$\Phi_r(0, \lambda) = 0$, $D_u\Phi_r(0, 0) = A_0|_U$, $x^*(0, \lambda) = 0$, and $D_u x^*(0, 0)\cdot \overline{u} = \overline{u}$, (for all $\overline{u} \in U$);*
2. *$\Phi_{r,\lambda}(S_0 u) = S_0\Phi_{r,\lambda}(u)$, $\forall\,(u, \lambda) \in U \times \mathbb{R}^m$ i.e. $\Phi_{r,\lambda}$ is $\mathbb{Z}_q$-equivariant;*
3. *for all sufficiently small $(x, \lambda) \in \mathbb{R}^{2n} \times \mathbb{R}^m$ we have that x is a q-periodic point of Φ_λ if and only if $x = x^*(u, \lambda)$ for some sufficiently small $u \in U$ which itself is a q-periodic point of $\Phi_{r,\lambda}$;*
4. *for all sufficiently small $(u, \lambda) \in U \times \mathbb{R}^m$ we have that u is a q-periodic point of $\Phi_{r,\lambda}$ if and only if*
$$\Phi_{r,\lambda}(u) = S_0 u, \tag{6}$$
i.e. all small q-periodic orbits of $\Phi_r(., \lambda)$ are necessarily $\mathbb{Z}_q$-orbits.

Moreover, if **(R)** *is satisfied we have*

5. $x^*(Ru,\lambda) = Rx^*(u,\lambda)$,
6. $R \circ \Phi_{r,\lambda} \circ R = \Phi_{r,\lambda}^{-1}$, *i.e.* $\Phi_{r,\lambda}$ *is* R*-reversible.*

This theorem establishes a one-to-one relation between the small q-periodic orbits of Φ_λ and those of the *reduced diffeomorphism* $\Phi_{r,\lambda}$ which lives on a *reduced space* U. This reduced diffeomorphism also retains the additonal structures of the original one, and, as we will explain in section 2, it can be approximated up to any order by using normal forms. Moreover, one can also prove the following.

Proposition 2. *For sufficiently small* $(u,\lambda) \in U \times \mathbb{R}^m$ *the equation (6) is equivalent to the equation*

$$\mathcal{B}(u,\lambda) := \Phi_{r,\lambda}\left(S_0^{-1}u\right) - S_0\Phi_{r,\lambda}^{-1}(u) = 0. \tag{7}$$

Observe that this equation is $\mathbb{D}_q$-equivariant: indeed, we have that

$$\mathcal{B}(S_0u,\lambda) = S_0\mathcal{B}(u,\lambda) \text{ and } \mathcal{B}(Ru,\lambda) = -R\mathcal{B}(u,\lambda), \tag{8}$$

with S_0 and R generating a $\mathbb{D}_q$-action on U, see properties (4).

2 Approximating the reduced mapping $\Phi_{r,\lambda}$

One of the possibilities to calculate the reduced mapping $\Phi_{r,\lambda}$ is to use normal form theory. The normal form techniques as well as the Lyapunov-Schmidt-like reductions are very popular tools for studying bifurcations. Here we provide a theorem on a reversible normal form for diffeomorphisms. More details and proofs can be found in [2]. These proofs are mainly inductive and are based on a combined use of the adjoint action of the group of diffeomorphisms satisfying **(H)** on itself and the implicit function theorem. Some technical results are of course needed to take care of the reversible structure[1]; to this purpose the following is a crucial Lemma.

Lemma 3. *Let* S_0 *be reversible and semisimple. Then there exists a scalar product* $\langle\cdot,\cdot\rangle$ *on* $\mathbb{R}^n$ *such that when we denote the transpose of a linear operator* $A \in GL_-(n,\mathbb{R})$ *with respect to this scalar product by* A^T *the following holds:*

(i) the involution R *is orthogonal, i.e.* $R^TR = I_{\mathbb{R}^n}$*;*

(ii) a linear operator $A \in \mathcal{L}(\mathbb{R}^n)$ *commutes with* S_0 *if and only if it commutes with* S_0^T*.*

Here we use $\langle x, Ay\rangle = \langle A^Tx, y\rangle$ for all $x, y \in \mathbb{R}^n$. For a detailed proof see [2] or [6] where a similar statement is proved when S_0 satisfies $S_0R = -RS_0$.
Denoting by the exponential the time-one map, one proves the following.

[1] The set of reversible diffeomorphisms does not form a Lie group.

Theorem 4. *For each $k \geq 1$ there exists a neighbourhood ω_k of the origin in $\mathbb{R}^m$ and a parameter dependent near identity transformation which brings Φ_λ in the form*

$$\Phi_\lambda = S_0 e^{\mathcal{N}_0 + X_\lambda} + R_{k+1}, \tag{9}$$

with $R_{k+1}(x,\lambda) = O\left(\|x\|^{k+1}\right)$ uniformily for $\lambda \in \omega_k$, and with X_λ a smooth parameter-dependent vectorfield on $\mathbb{R}^n$ such that $X_\lambda(0) = 0$, $DX_0(0) = 0$ and

$$X_\lambda(S_0 x) = S_0 X_\lambda(x), \quad e^{t\mathcal{N}_0^T} X_\lambda = X_\lambda e^{t\mathcal{N}_0^T}. \tag{10}$$

Moreover, the vectorfield X_λ is reversible:

$$X_\lambda(Rx) = -RX_\lambda(x). \tag{11}$$

Then we call $\Phi_\lambda^{NF} = S_0 e^{\mathcal{N}_0 + X_\lambda}$ the normal form of Φ_λ up to order k.
This normal form can be used to approximate the reduced mapping $\Phi_{r,\lambda}$: if Φ_λ is in normal form up to order k then

$$\Phi_{r,\lambda}(u) = \Phi_\lambda^{NF}(u) + O\left(\|u\|^{k+1}\right) \text{ and } x^*(u,\lambda) = u + O\left(\|u\|^{k+1}\right).$$

Just to give an idea on how to deal with the reversibility in normal forms, we prove Theorem 4 in the case of linear reversible operators. We use the following notations

$$GL_\pm(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) |\, RAR = A^{\pm 1}\}$$
$$gl_{\pm R}(n,\mathbb{R}) := \{A \in gl(n,\mathbb{R}) |\, RA = \pm AR\}.$$

Notice that $gl_{+R}(n,\mathbb{R})$ is a Lie algebra and the corresponding Lie group is then $GL_+(n,\mathbb{R})$. Also, for $\Psi \in GL(n,\mathbb{R})$ define the operator $Ad(\Psi) : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ by

$$Ad(\Psi)\Phi := \Psi \cdot \Phi \cdot \Psi^{-1}, \quad \forall \Phi \in GL(n,\mathbb{R}), \tag{12}$$

and for $\psi \in gl(n,\mathbb{R})$ define $ad(\psi) \in \mathcal{L}\big(gl(n,\mathbb{R})\big)$ by

$$ad(\psi)A = \psi A - A\psi, \quad \forall A \in gl(n,\mathbb{R}). \tag{13}$$

Observe that for each $\Psi \in GL(n,\mathbb{R})$ the automorphism $Ad(\Psi)$ on $GL(n,\mathbb{R})$ induces a linear mapping on $gl(n,\mathbb{R})$ obviously given by

$$Ad(\Psi) \cdot A := \Psi \cdot A \cdot \Psi^{-1}, \quad \forall A \in gl(n,\mathbb{R}).$$

Notice also that if $\Psi \in GL_{+R}(n,\mathbb{R})$ then $Ad(\Psi) : gl_\pm(n,\mathbb{R}) \to gl_{\pm R}(n,\mathbb{R})$, while if $\psi \in gl_{-R}(n,\mathbb{R})$ then $ad(\psi) : gl_{\pm R}(n,\mathbb{R}) \to gl_{\mp R}(n,\mathbb{R})$. We start with the following consequence of Lemma 3, which is crucial in the proof Theorem 4.

Corollary 5. *Let $A_0 = S_0 e^{\mathcal{N}_0}$ be the SU-decomposition of $A_0 \in GL_-(n,\mathbb{R})$ and let $\langle\cdot,\cdot\rangle$ be a scalar product as in as in Lemma 3. Then also A_0^T, S_0^T belong to*

$GL_-(n,\mathbb{R})$, and ${\mathcal{N}_0}^T$ belongs to $gl_{-R}(n,\mathbb{R})$. Moreover, the following direct sum splitting holds:

$$\ker\big(Ad(S_0)-I\big)\cap gl_{-R}(n,\mathbb{R}) = \Big[ad\mathcal{N}_0\Big(gl_{+R}(n,\mathbb{R})\cap\ker(Ad(S_0)-I)\Big)\Big] \oplus\Big[\ker\big(Ad(S_0)-I\big)\cap gl_{-R}(n,\mathbb{R})\cap\ker ad\left(\mathcal{N}_0^T\right)\Big]. \tag{14}$$

For the proof we again refer to [2]. We have now all the ingredients to prove the main result on normal forms of linear reversible operators.

Proposition 6. *Let $A_0 = S_0 e^{\mathcal{N}_0}$ be the SU-decomposition of a given R-reversible operator on $\mathbb{R}^n$. Then there exist a neighbourhood U of A_0 in $GL_-(n,\mathbb{R})$ and a mapping $\Psi : U \to GL_+(n,\mathbb{R})$ such that*

$$\Psi(A_0) = I \quad \text{and} \quad Ad\big(\Psi(A)\big)\cdot A = S_0 e^{\mathcal{N}_0 + B(A)}, \quad \forall A\in U, \tag{15}$$

with $B(A_0)=0$ and $B(A)\in\ker\big(\mathrm{Ad}(\mathrm{S}_0)-\mathrm{I}\big)\cap\ker\big(ad(\mathcal{N}_0^T)\big)\cap gl_-(n,\mathbb{R})$, i.e. $B(A)$ commutes with S_0 and $\mathcal{N}_0^T$.

Proof. Consider the direct sum splitting (14) and denote by π the linear projection of $\ker\big(Ad(S_0)-I\big)\cap gl_{-R}(n,\mathbb{R})$ onto the first component and parallel to the second component. Referring to a previous normal form result in [3], (see also [2]), we may assume the existence of a neighbourhood V of A_0 in $GL_-(n,\mathbb{R})$ such that all $A\in V$ can, via an appropriate near identity transfomation, be written in the form $A = S_0 e^{C(A)}$, for some smooth

$$C : V \to \ker\big(Ad(S_0)-I\big)\cap gl_{-R}(n,\mathbb{R})$$

such that $C(A_0)=\mathcal{N}_0$. Knowing that if $\Psi\in\ker\big(Ad(S_0)-I\big)\cap GL_+(n,\mathbb{R})$ then

$$Ad(\Psi)\cdot(A) = S_0 e^{Ad(\Psi)\cdot C(A)},$$

with $Ad(\Psi)\cdot C(A)\in\ker\big(Ad(S_0)-I\big)\cap gl_{-R}(n,\mathbb{R})$, then we only have to determine Ψ, dependent on A, such that

$$Ad(\Psi)\cdot C(A) = \mathcal{N}_0 + B,$$

with $B\in\ker\big(\mathrm{Ad}(\mathrm{S}_0)-\mathrm{I}\big)\cap\ker\big(ad(\mathcal{N}_0^T)\big)\cap gl_-(n,\mathbb{R})$. To do so we define a mapping

$$F : \Big[\ker\big(Ad(S_0)-I\big)\cap gl_{+R}(n,\mathbb{R})\Big]\times V \to ad\mathcal{N}_0\Big(\ker\big(Ad(S_0)-I\big)\cap gl_{+R}(n,\mathbb{R})\Big)$$

by

$$F(\varphi,A) := \pi\big(Ad(\Psi)\cdot C(A)-\mathcal{N}_0\big).$$

It follows that

$$F(I,A_0)=0 \quad \text{and} \quad D_\Psi F(I,A_0) = \pi\cdot ad(\mathcal{N}_0)|_{\ker\left(Ad(S_0)-I\right)\cap gl_{+R}(n,\mathbb{R})}.$$

The linear operator $D_{\Psi}F(I,A_0)$ is surjective from $\ker\big(Ad(S_0)-I\big)\cap gl_{+R}(n,\mathbb{R})$ onto $ad\mathcal{N}_0\Big(gl_{+R}(n,\mathbb{R})\cap\ker\big(Ad(S_0)-I\big)\Big)$. Hence we can invoke the Implicit Function Theorem to conclude that there exist a neighbourhood $U\subset V\subset GL_-(n,\mathbb{R})$ of A_0 and a smooth mapping $\Psi: U\to\ker\big(Ad(S_0)-I\big)\cap GL_{+R}(n,\mathbb{R})$ with $\Psi(A_0)=I$ and such that $F(\Psi(A),A)=0$, for all $A\in U$. By definition of F and π it follows that

$$B(A):=Ad\big(\Psi(A)\big)\cdot C(A)-\mathcal{N}_0\in\ker\big(Ad(S_0)-I\big)\cap\ker\big(ad(\mathcal{N}_0^T)\big)\cap gl_{-R}(n,\mathbb{R}),$$

which proves the proposition.

3 Generic bifurcation of periodic points

In this section we show how the foregoing reduction can be used to describe a simple type of bifurcation which occurs generically when studying branching of solutions at a symmetric periodic solution of an autonomous time-reversible system. To this purpose, consider a $2k$-dimensional autonomous time-reversible vector field such that dim $Fix(R)=k$. Assume that the system has a non-constant R-symmetric periodic solution γ_0 with period T_0. We are interested in other periodic orbits of the system nearby the given γ_0. In order to study such orbits we consider the Poincaré map Φ associated to γ_0. It turns out that Φ is a local diffeomorphism satisfying **(H)** and **(R)** with $n=2k-1$ and $m=0$. Cfr. [2,5]. The fixed point 0 of Φ corresponds to γ_0, other fixed points correspond to periodic orbits of the system close to γ_0 with minimal period close to T_0. Finally, q-periodic orbits of Φ correspond to so-called *subharmonic* solutions of the system, that is, periodic orbits near γ_0 with minimal period near qT_0.

We first study the periodic orbits near γ_0 with minimal period near T_0 by looking for fixed points of Φ near 0 (i.e. $q=1$ in **(P)**). Assume the simplest possible situation:

(H1) the operator A_0 has $+1$ as simple eigenvalue.

It follows that $\ker(S_0-I)$ is one-dimensional, moreover $Ru=u$, for all $u\in\ker(\mathrm{S}_0-\mathrm{I})$, [2,3]. Then via the reduction we obtain the following result.

Theorem 7. *Assume* **(H)**, **(R)**, **(H1)**. *Then there exists a smooth mapping* $x^*: ker(S_0-I)\to\mathbb{R}^n$ *such that*

a) $x^*(0)=0$;
b) $Rx^*(u)=x^*(u),\ \forall u\in ker(S_0-I)$;
c) $\Phi(x^*(u))=x^*(u)$, *for all sufficiently small* $u\in ker(S_0-I)$.

Moreover, Φ *has in some neighbourhood of the origin no other fixed points than those on the curve* $\{x^*(u)\,|\,|u|<u_0\}$.

We shall call this branch of fixed point the *primary branch*. The matrix $D\Phi\left(x^*(u)\right)$ is reversible, and hence if $\mu \in \mathbb{C}$ is an eigenvalue, so is μ^{-1}. It follows that if $D\Phi\left(x^*(u)\right)$ has for $u = u_0$ a pair of simple eigenvalues on the unit circle, these eigenvalues will stay on the unit circle for all u near u_0. Assuming that they move with non-zero speed it follows that along the primary branch one may find symmetric fixed points at which the linearization has eigenvalues which are q-th roots of unity, for some $q \geq 3$. As shown in the next paragraph, this situation leads to branching of periodic points for Φ, which means subharmonic branching for the original system.

Take $q \geq 3$ in **(P)** and assume the following

(H2)- A_0 has a pair of simple eigenvalues $\left(\chi_q, \overline{\chi}_q\right)$, with $\chi_q = \exp\left(2i\pi\frac{p}{q}\right)$ and $q \geq 3$, $0 < p < q$, $\gcd(p,q) = 1$;
- A_0 has, besides 1, χ_q and $\overline{\chi}_q$, no other eigenvalues $\mu \in \mathbb{C}$ such that $\mu^q = 1$.

One can easily see that the continuation of the eigenvalue χ_q can be written as $e^{i\alpha_q(\lambda)}\chi_q$, with $\alpha_q(\lambda) \in \mathbb{R}$ and $\alpha_q(0) = 0$. More precisely, the eigenvalues will move along the unit circle as we move along the primary branch. We assume the transversality condition

(T) $\alpha_q'(\lambda) \neq 0$.

Notice that $\dim\ker\left(S_0^q - I\right) = \dim U = 3$ and that $\ker\left(S_0 - I\right) \subset \ker\left(S_0^q - I\right)$. Also, denoting by V the S_0-invariant complement of $\ker\left(S_0 - I\right)$ in U, we can identify U with the direct product $\mathbb{R} \times \mathbb{C}$, where $\ker\left(S_0 - I\right) \cong \mathbb{R}$ and $V \cong \mathbb{C}$. Moreover, $S_0|_{\mathbb{C}}$ acts as multiplication by χ_q and $R|_{\mathbb{C}}$ acts as $z \mapsto \overline{z}$. It follows that the reduced bifurcation mapping (7) on $U = \mathbb{R} \times \mathbb{C}$ takes the form:

$$\mathcal{B}(\alpha, z) = \big(b_0(\alpha, z), b_1(\alpha, z)\big), \tag{16}$$

with $b_0 : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and $b_1 : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned}
&b_0(0,0) = 0, \quad b_1(0,0) = 0 \\
&b_0(\alpha, \chi_q z) = b_0(\alpha, z), \quad b_1(\alpha, \chi_q z) = \chi_q b_1(\alpha, z) \\
&b_0(\alpha, \overline{z}) = -b_0(\alpha, z), \quad b_1(\alpha, \overline{z}) = -\overline{b_1(\alpha, z)}.
\end{aligned}$$

Compare with [2,3]. Using a result on the normal form of complex $\mathbb{D}_q$-equivariant functions, see [4,1], it follows that the non-trivial solutions of the bifurcation equation not lying on the primary branch are the solutions of

$$b_1(\alpha, z) = i\theta_1\left(\alpha, z\right) z + i\theta_2\left(\alpha, z\right) \overline{z}^{q-1} = 0 \tag{17}$$

with $\theta_i : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ $(i = 1, 2)$ smooth, real-valued and $\mathbb{D}_q$-invariant functions. Using polar coordinates and some generically satisfied conditions one obtains the existence of exactly two R-symmetric branches of q-periodic orbits bifurcating from the fixed point-branch. These branches have the form

$$\gamma_i = \left\{ \left(\widetilde{\alpha}_i\left(\rho\right)\right), \chi_q^j \widetilde{z}_i\left(\rho\right) \middle| \, 0 < \rho < \rho_0, 0 \leq j \leq q-1 \right\}, \; (i = 1, 2) \tag{18}$$

with $\widetilde{z}_1(\rho) := \rho$, $\widetilde{z}_2(\rho) := \rho e^{i\frac{\pi}{q}}$, while the functions $\alpha = \widetilde{\alpha}_i(\rho)$ are the solutions of the equation

$$\theta_1\big(\alpha, \widetilde{z}_i(\rho)\big) + (-1)^i \rho^{q-2}\theta_2\big(\alpha, \widetilde{z}_i(\rho)\big) = 0 \quad (i = 1, 2), \tag{19}$$

such that $\widetilde{\alpha}_i(0) = 0$. Setting $\widetilde{x}_i(\rho) := x^*\big(\widetilde{\alpha}_i(\rho), \widetilde{z}_i(\rho)\big)$ gives then two branches of q-periodic points of Φ bifurcating from the fixed point, since $\big(\widetilde{\alpha}_i(0), \widetilde{x}_i(0)\big) = (0, 0)$. We conclude with some remarks:

(a) the greater is q the closer are the two branches in parameter space. Indeed, one can show that $|\widetilde{\alpha}_1(\rho) - \widetilde{\alpha}_2(\rho)| = O\left(\rho^{\frac{q-2}{2}}\right)$;

(b) if $\theta_2(0, 0) \neq 0$ there are no other q-periodic orbits (close to $(0, 0)$) than the two branches we have found.

For similar bifurcation results in the symplectic case we refer to [4].

4 Stability

In this section we show how one can obtain some information on the stability of bifurcating periodic orbits. When $x \in \mathbb{R}^n$ generates a q-periodic orbit of Φ_λ then the stability of this orbit is determined by the eigenvalues of $D\Phi_\lambda^q(x)$: the orbit is stable if all eigenvalues are inside the unit circle, and unstable if some eigenvalues are outside the unit circle. When the periodic orbit is symmetric (i.e. invariant under R) then together with $\mu \in \mathbb{C}$ also μ^{-1} will be an eigenvalue of $D\Phi_\lambda^q(x)$; in such case the orbit will be unstable if some eigenvalue is off the unit circle, and there will be a weak form of stability if all eigenvalues are on the unit circle. When applying this to bifurcating periodic orbits we have to determine the eigenvalues of $D\Phi_\lambda^q(x^*(u, \lambda))$ for all small $(u, \lambda) \in U \times \mathbb{R}^m$ which satisfy (6). For $(u, \lambda) = (0, 0)$ this operator reduces to A_0^q, which has eigenvalues 1 on U and eigenvalues away from 1 on $V := \operatorname{Im}(S_0^q - I)$; it follows that for small (u, λ) the operator $D\Phi_\lambda^q(x^*(u, \lambda))$ will have some eigenvalues near 1, with total multiplicity equal to $\dim U$, and all other eigenvalues uniformly bounded away from 1. We call the eigenvalues near 1 the critical eigenvalues. If the non-critical eigenvalues of A_0^q are all simple and on the unit circle then the critical eigenvalues of $D\Phi_\lambda^q(x^*(u, \lambda))$ will determine the stability of the corresponding periodic orbit. To approximate these critical eigenvalues, define a smooth mapping $\mathcal{D} : U \times \mathbb{R}^m \to L(\mathbb{R}^n)$ by

$$\mathcal{D}(u, \lambda) := D\Phi_\lambda\left(x^*\left(S_0^{q-1}u, \lambda\right)\right) \cdots D\Phi_\lambda\left(x^*(S_0 u, \lambda)\right) \cdot D\Phi_\lambda\left(x^*(u, \lambda)\right). \tag{20}$$

When (u, λ) satisfies (6) then $\Phi_\lambda\big(x^*(u, \lambda)\big) = x^*(S_0 u, \lambda)$ and $D\Phi_\lambda^q(x^*(u, \lambda)) = \mathcal{D}(u, \lambda)$; therefore it is sufficient to determine the critical eigenvalues of $\mathcal{D}(u, \lambda)$ for such (u, λ). As an example, it turns out that the stability of the symmetric periodic orbits along the branch γ_i (see (18)), is determined by the number

$$\tau_i(\rho) := \operatorname{tr} \widetilde{\mathcal{D}}\left(\widetilde{z}_i(\rho), \widetilde{\lambda}_i(\rho)\right), \tag{21}$$

where

$$\widetilde{\mathcal{D}}\left(\widetilde{u}(\rho), \widetilde{\lambda}(\rho)\right) = \exp\left(DX_{\widetilde{\lambda}(\rho)}\left(\widetilde{u}(\rho)\right)\right) + O\left(\rho^k\right); \tag{22}$$

see [2,3] for the details. The calculations show that one of the two branches is stable, the other is unstable, (cfr. [2,3]).

5 1:1 resonance: some remarks

Returning to the situation described in section 3 the following scenario can happen. The mapping Φ has a fixed point at which the linearization of Φ has two pairs of simple complex conjugate eigenvalues on the unit circle close to each other. Then, when moving along the corresponding primary branch, phenomena such as 'collision' and 'splitting' may happen, that is: the eigenvalues collide into a pair of non-semisimple double eigenvalues on the unit circle and then plitt off the circle. Introducing an external parameter $\lambda \in \mathbb{R}$, one may arrange things such that for some value of the parameter, $\lambda = \lambda_0$, the collision happens at a q-th root of unity. It is then natural to ask what kind of bifurcation scenario emerges when solving problem **(P)** for a one-parameter family of reversible mappings satisfying **(H)**, **(R)** and the following:

(H3)- A_0 has simple eigenvalue 1 and eigenvalues $\left(\chi_q, \overline{\chi}_q\right)$ with algebraic multiplicity 2 (and geometric multiplicity 1),
- A_0 has no other eigenvalues on the unit circle.

Application of the reduction result of section 1 shows that we are left with a 5-dimensional problem on U, $\dim U = 5$. Also in this case $\ker\left(S_0 - I\right) \subset U$ can be identified with $\mathbb{R}$ and its 4-dimensional S_0-invariant complement in U with $\mathbb{C} \times \mathbb{C}$. So the reduced bifurcation equation consists of one real equation and two complex ones. A combined use of the Normal Form Theorem 4 and Implicit Function Theorem allows us to solve one of the two complex equation; we are then left with a problem similar to that of subharmonic branching analized in section 3. In a forthcoming paper we will describe the full bifurcation picture.

References

1. Maria-Cristina Ciocci. *Subharmonic branching in Hamiltonian systems.* Master thesis University "La Sapienza" of Rome, 1997.
2. Maria-Cristina Ciocci. *PhD Thesis,* in preparation. University of Gent.
3. Maria-Cristina Ciocci. *On the stability of bifurcating periodic orbits in reversible and symplectic mappings,* Proceedings Equadiff '99 Berlin, World Scientific (2000) vol.1, pag. 166-168.
4. M.C. Ciocci, A. Vanderbauwhede, University of Gent . *Bifurcation of periodic orbits for symplectic mappings.* Journal of Difference Equations and Applications, 1998.

5. J. Knobloch and A. Vanderbauwhede. *A general reduction method for periodic solutions in consrvative and reversible systems.* Journal of Dynamics and Diff. Eqns., 8 (1996) 71-102.
6. J. Knobloch and A. Vanderbauwhede. *Hopf bifurcation at k-fold resonances in reversible systems.* Preprint TechnischeUniversit at Ilmenau, No. M 16/95.
7. A.Vanderbauwhede. *Normal forms and versal unfoldings of symplectic linear maps.* World Scientific Ser.in Appl. Anal. 4 (1995) 685-700.
8. A.Vanderbauwhede. *Subharmonic bifurcation at multiple resonances.* Proceedings of the Mathematics Conference, Birzeit University, Palestine 1998. World Scientific (2000) 254-276
9. A.Vanderbauwhede. *Hopf bifurcation for equivariant conservative and time-reversible systems.* Proceedings of the Royal Society of Edimburgh, 116A, 103-128, 1990.
10. A.Vanderbauwhede. *Bifurcation of subharmonic solutions in time-reversible systems.* Journal of Applied Mathematics and Physics, vol 37, July 1986.
11. A.Vanderbauwhede. *Stability of bifurcating solutions.* Publications de l'UER Mathematiques Pures et Appliques, Université de Lille, vol.5 (2)(1983).