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System of Differential Equations with Unstable Turning Point and Multiple Element of Spectrum of Degenerate Operator

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Abstract. A uniform asymptotics of a solution is constructed for a system of singularly perturbed differential equations with a turning point. The paper investigates the case when the spectrum of a phase operator consists of multiple elements.

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1 Introduction

The systems of singular perturbation differential equations (SSPDE) play the great role in many mathematical models of biological problems and in medicine. This can be seen from monograph [1] in which kinetic rule of Michaelis-Menten is described by Tikhonov SSPDE (see equations (1.20), (1.21)). One of the known problems for investigation such problems is as follows: Is there a biological process stable or not? If the spectrum of the degenerate operator is stable, then such a system is known pretty well. Remembering classical results (Vasilieva's method, Lomov's method and others) which give the answer of the discussed problems concerning bounded solutions of the adequate SSPDE. If the system contains turning points, i.e. some elements of the spectrum of the degenerate operator are unstable, then the general theory of investigating such problems is not constructed yet, however some special problems have the solution (see [2,3,4]).

In the presented article we consider the problem:

$$\left.\begin{aligned} L_\varepsilon W(x,\varepsilon) &\equiv \varepsilon^2 W''(x,\varepsilon) - AW(x,\varepsilon) = h(x)\,,\\ E_1 W(m,\varepsilon) &= E_1(\mu^{-2}\alpha_{m.} + \widehat{W}_m)\,,\\ E_2 \frac{dW(m,\varepsilon)}{dx} &= E_2(\mu^{-4}\alpha_{m.} + \mu^{-3}\widehat{W}_m)\,, \end{aligned}\right\} \tag{1}$$

This is the final form of the paper.

where

$$\varepsilon \to +0, \qquad x \in I = [0,1], \qquad m = 0,1, \qquad \mu = \sqrt[3]{\varepsilon}\,.$$

Here A denotes a linear operator on R^n, α_m and $\widehat{W}_m$ are given vectors, E_k $(k = 1,2)$ — diagonal matrices of the nth order of the form $E_1 = \mathrm{diag}\{1,0,\dots,0\}$, $E_2 = \mathrm{diag}\{0,1,\dots,1)$, $h(x)$ — given vector-function, $W(x,\varepsilon)$ — a sought vector-function.

We shall consider the problem (1) under the following conditions:

1° $A(x),\ h(x) \in C^\infty[I]$

2° Spectrum of the degenerate operator A is real and fulfills the following condition

$$0 \le \lambda_1(x) \equiv x\widetilde{\lambda}_1(x) < \lambda_2(x) < \cdots < \lambda_{p.}(x) \equiv \cdots \equiv \lambda_n(x), \tag{2}$$

where $\widetilde{\lambda}_1(x) > 0$ for all $x \in I$.

One can see from (2) that the point $x = 0$ is a turning point for equation (1), and moreover of $\lambda_1(x) \ge 0$, then it is unstable turning point. Thus it is necessary to construct a solution of (1) in the case when the simplified equation

$$-A\omega(x) = h(x) \tag{3}$$

has, in general case, a point of discontinuity at 0. System (1) with several conditions for the spectrum of degenerate operator A was consider in [2,3,4,5] and in other articles of authors.

In the present article we shall prove, that ignoring the unstability of the turning point some partial solutions of the vector equation (1) can be bounded in the domain.

2 Extension of the perturbation problem

One of the main problem of asymptotics of solution of singular perturbed problem (SPP) (1) is: choice, description and conserve them as a whole all essentially singular manifolds (ESM) contained in the solutions SPP (1).

For this purpose, together with the independent variable x, we shall consider vector-variable

$$\left.\begin{aligned} &t = \{t_{ik}\}\,, \quad i = \overline{1,p}\,, \quad k = 1,2 \quad \text{according to}\\ &t_{1k} \equiv t_1 = \mu^{-2}\left(\frac{3}{2}\int_0^x \sqrt{\lambda_1(x)}dx\right)^{\frac{2}{3}} \equiv \mu^{-2}\varphi_1(x) \equiv \phi_1(x,\varepsilon),\\ &t_{jk} = \mu^{-3}(-1)^k \int_{(k-1)l}^x \sqrt{\lambda_j(x)}dx \equiv \mu^{-3}\varphi_{jk}(x) \equiv \phi_{jk}(x,\varepsilon),\\ &\quad k = 1,2, \quad j = \overline{2,p} \end{aligned}\right\} \tag{4}$$

Then instead of vector-function $W(x,\varepsilon)$ we shall consider a new "extended" vector-function $\widetilde{W}(x,t,\varepsilon)$, where in view of the regularization method (see [5]), the extension is taken in such a way that

$$\widetilde{W}(x,t,\varepsilon)\Big|_{t=\phi(x,\varepsilon)} \equiv W(x,\varepsilon), \tag{5}$$

where

$$\phi(x,\varepsilon) = \{\phi_1(x,\varepsilon), \phi_{jk}(x,e),\ k=1,2;\ j=\overline{2,p}\}\,.$$

Differentiating twice the identity (5) and substituting the derivative of the second order of $\widetilde{W}(x,t,\varepsilon)$ to equation (1), we obtain, for the extension $\widetilde{W}(x,t,\varepsilon)$ the following "extended" problem:

$$\left.\begin{aligned} \widetilde{L}_\varepsilon \widetilde{W}(x,t,\varepsilon) &= h(x)\,, \qquad M_m = (m, t(m))\,, \\ E_1\widetilde{W}(M_m,\varepsilon) &= E_1(\mu^{-2}\alpha_m + \widehat{W}_m)\,, \\ E_2\frac{dW(M_m,\varepsilon)}{dx} &= E_2(\mu^{-4}\alpha_m + \mu^{-3}\widehat{W}_m)\,. \end{aligned}\right\} \tag{6}$$

Here

$$\widetilde{L}_\varepsilon \equiv \mu^2\varphi_1'^2(x)\frac{\partial^2}{\partial t_1^2} + \mu^4 d_1\frac{\partial}{\partial t_1} + \mu^6\frac{\partial^2}{\partial x^2} - \\ - A + \sum_{k=1}^{2}\sum_{j=2}^{p}[\varphi_{jk}'^2(x)\frac{\partial}{\partial t_{jk}} + \mu^3 d_{jk}]\frac{\partial}{\partial t_{jk}} + Y_\varepsilon^1\,, \tag{7}$$

where

$$d_{jk} \equiv 2\varphi_{jk}'(x)\frac{\partial}{\partial x} + \varphi_{jk}''(x)\,. \tag{8}$$

Y_ε^1 — operator that plays the role of annihilator. Then there is no idea to give its full form.

3 Spaces of nonresonance solutions

We shall describe the sets (subspaces) of functions in which we shall solve the extended problem (6). We have

$$\left.\begin{aligned} Y_{rilk} &= \{b_i(x)[V_{rik}(x)U_k(t_1) + Q_{rik}(x)U_k'(t_1)]\}\,, \\ Y_{rijk} &= \{b_i(x)\alpha_{rijk}(x)\exp(t_{jk})\},\ j=\overline{2,p}\,, \\ V_{ri} &= \{b_i(x)[f_{ri}(x)\nu(t_1) + g_{ri}(x)\nu'(t_1)]\}\,, \\ X_{ri} &= \{b_i(x)\omega_{ri}(x)\}\,,\ i=\overline{1,n};\ k=1,2, \end{aligned}\right\} \tag{9}$$

where the coefficients in ESM and the functions $\omega_{ri}(x)$ are arbitrary sufficiently smooth functions for x in I, and $b_i(x)$ $(i=\overline{1,n})$ — a complete system of proper vectors for proper values $\lambda_i(x)$ $(i=\overline{1,p})$. Since the operator A is operator of a simple structure, then for a multiple proper value $\lambda_p(x)\equiv\cdots\equiv\lambda_n(x)$ there exists a system of linearly independent vectors (fundamental solutions) $b_i(x)$ $(i=\overline{p,n})$.

By $b_i^*(x)$ $(i=\overline{1,n})$ we shall denote the complete system of proper vectors of the conjugate operator A^*, where those vectors are chosen in such a way that together the vectors $b_i(x)$ they form a biorthogonal system of vectors. The existence of such a system is proved since A is an operator of a simple structure (see [6, p. 218]).

Later on $U_1(t_1)\equiv Ai(t_1)$, $U_2(t_1)\equiv Bi(t_1)$ — are the Eiry-functions, which properties are described in monograph [7, chapter 1.1].

Essentially singular manifolds $\nu(t_1)\equiv -G_i(t_1)$ — are Skorera functions (see [7, p. 412]).

From the subspaces (9) we shall construct the space of the form:

$$Y_r=\bigoplus_{i=1}^{n} Y_{ri}\equiv\bigoplus_{i=1}^{n}\Big[\bigoplus_{k=1}^{2}\bigoplus_{j=1}^{p}Y_{rijk}\oplus V_{ri}\oplus X_{ri}\Big]\,, \tag{10}$$

which in view of the known terminology [5] will be called the space of nonresonance solutions (SNS).

The element $W_r(x,t)\in Y_r$ has the form

$$W_r(x,t)=\sum_{i=1}^{n}b_i(x)W_{ri}(x,t)\equiv\sum_{i=1}^{n}\widetilde{W}_{ri}(x,t)\,, \tag{11}$$

where

$$W_{ri}(x,t)\equiv\sum_{k=1}^{2}\Big[V_{rik}(x)U_k(t_1)+Q_{rik}(x)U_k'(t_1)+ \\ +\sum_{j=2}^{p}\alpha_{rijk}(x)\exp(t_{tj})\Big]+f_{ri}(x)\nu(t_1)+g_{ri}(x)\nu'(t_1)+\omega_{ri}(x)\,.$$

4 Regularization of the singular perturbed problem

We have to find the properties of the extended operator $\widetilde{L}_\varepsilon$ on elements from SNS (10). Method of obtaining such procedure is described in articles [2,3,4,5]. That is why we shall write only the final result of that property. We have:

$$\widetilde{L}_\varepsilon W_r(x,t)\equiv[R_0+\mu^2R_2+\mu^3R_3+\mu^4R_4+\mu^6R_6]W_r(x,t)\,. \tag{12}$$

Operators R_s can be written in the form

$$R_0 W_r(x,t) \equiv \sum_{i=1}^{n} b_i(x)\Big\{(\lambda_1-\lambda_i)\Big[\sum_{k=1}^{2}[V_{rik}(x)U_k(t_1)+Q_{rik}(x)U_k'(t_1)] + \\ + f_{ri}(x)\nu(t_1)+g_{ri}(x)\nu'(t_1)\Big] + \\ + \sum_{j=2}^{p}(\lambda_j-\lambda_i)\sum_{k=1}^{2}\alpha_{rijk}(x)\exp(t_{jk}) - \lambda_i(x)\omega_{ri}(x)\Big\}, \tag{13}$$

$$R_2 W_r(x,t) \equiv \sum_{i=1}^{n} b_i(x)\Big\{\Big[\sum_{s=1}^{n}\varphi_1\cdot T_{si1}+\widetilde{D}_{i1}\Big]\times \\ \times\Big[\sum_{k=1}^{2}Q_{rik}(x)U_k(t_1)+g_{ri}(x)\nu(t_1)\Big] - \pi^{-1}\varphi_1'^2(x)f_{ri}(x)\Big\}, \tag{14}$$

$$R_3 W_r(x,t) \equiv \sum_{i=1}^{n} b_i(x)\Big\{\sum_{j=2}^{p}\sum_{k=1}^{2}\Big[D_{ijk}\alpha_{rijk}(x)+\sum_{\substack{s\neq i\\ s=1}}^{n}T_{sijk}\alpha_{rsjk}(x)\Big]\exp(t_{jk})\Big\}, \tag{15}$$

$$R_4 W_r(x,t) \equiv \sum_{i=1}^{n} b_i(x)\Big\{\Big[\sum_{\substack{s\neq i\\ s=1}}^{n}T_{si1}+D_{i1}\Big]\cdot\Big[\sum_{k=1}^{2}V_{rik}(x)U_k'(t_1)+ \\ + f_{ri}(x)\nu'(t_1)+\pi^{-1}g_{ri}(x)\Big]\Big\}, \tag{16}$$

$$R_6 W_r(x,t) \equiv \frac{\partial^2\widetilde{W}_r(x,t)}{\partial x^2}. \tag{17}$$

Here

$$D_{ijk} \equiv 2\varphi_{jk}'(x)\left[\frac{\partial}{\partial x}+(b_i'(x),b_i^*(x))\right], \\ \widetilde{D}_{i1} \equiv \varphi_1(x)D_{i1}+\varphi_1'^2(x) \equiv \varphi_1(x)\cdot 2\varphi_1'(x)\left[\frac{\partial}{\partial x}+(b_i'(x),b_i^*(x))\right]+\varphi_1'^2(x), \\ T_{si1} \equiv 2\varphi_1'(x)(b_s'(x),b_i^*(x)), \quad T_{sijk} \equiv 2\varphi_{jk}'(x)(b_s'(x),b_i^*(x)). \tag{18}$$

Analyzing the obtained identities we can make the following implications.

1. Spaces of nonresonance solutions Y_r are invariant with respect to operators R_0, R_2, R_3, R_6 and consequently with respect to the extended operator $\widetilde{L}_\varepsilon$ which is represented in the form (12).
2. Operator R_0 is a main operator of the extended operator $\widetilde{L}_\varepsilon$ in SNS (10).
3. Extended problem (6) depends regularly on a small parameter $\mu > 0$ in SNS (10).

5 Formalism of construction of solution of extended problem

Since the extended problem (6) is regularly dependent on a small parameter $\mu > 0$ in SNS (10), then the asymptotic solution of that problem can be found in the form of a series

$$\widetilde{W}(x,t,\varepsilon) = \sum_{r=-2}^{+\infty} \mu^r W_r(x,t)\,, \tag{19}$$

where

$$W_r(x,t) \in Y_r\,.$$

Let us substitute (19) to the extended problem (6) and compare the coefficients by the parameter $\mu > 0$. Then for defining of the coefficients of the series (19) we shall get the following recurrence system of problems:

$$\begin{aligned} &R_0 W_{-2}(x,t) = 0, \quad E_1 W_{-2}(M_m) = E_1 \alpha_m\,, \\ &G_m W_{-2}(x,t) \equiv E_2 \sum_{k=1}^{2}\sum_{j=2}^{p} \varphi'_{jk}(m) \frac{\partial W_{-2}(M_m)}{\partial t_{jk}} = 0\,, \end{aligned} \tag{20}$$

$$\begin{aligned} &R_0 W_{-1}(x,t) = 0, \quad E_1 W_{-1}(M_m) = 0\,, \\ &G_m W_{-1}(x,t) = E_2 \Big[\alpha_m - \varphi'_1(m) \frac{\partial W_{-2}(M_m)}{\partial t_1}\Big]\,, \end{aligned} \tag{21}$$

$$\begin{aligned} &R_0 W_0(x,t) = h(x) - R_2 W_{-2}\,, \quad E_1 W_0(M_m) = E_1 \widehat{W}_m\,, \\ &G_m W_0(x,t) = E_2 \Big[\widehat{W}_m - \varphi'_1(m) \frac{\partial W_{-1}(M_m)}{\partial t_1}\Big]\,, \end{aligned} \tag{22}$$

In this way we obtain a series of recurrence problems with point boundary conditions which is in the general case insufficient for uniqueness of solutions of each of those separate problems. However, further considerations show that in SNS (10) the series of problems (20)–(22) is asymptotically correct, i.e. each of the problems (20)–(22) has the unique solution in SNS if we consider those problems step by step. Further on we should ask a question of the existence of a solution in SNS (10) of the iteration equation

$$R_0 W_r(x,t) = H_r(x,t)\,. \tag{23}$$

From identity (13) one can describe the structure of the kernel of the operator R_0. We have

$$\begin{aligned} \mathrm{Ker} R_0 = \big\{ & b_j(x)\alpha_{rjjk}(x)\exp(t_{jk}), j = \overline{2, p-1}, \\ & b_j(x)\alpha_{rjjk}(x)\exp(t_{pk}), j = \overline{p, n}\,, \\ & b_1(x)[V_{r1k}(x)U_k(t_1) + Q_{r1k}(x)U'_k(t_1)], \\ & b_1(x)[f_{r1}(x)\nu(t_1) + g_{r1}(x)\nu'(t_1)], k = 1, 2\big\} \end{aligned} \tag{24}$$

We introduce the following notations: $b_i(x)S_{ri}(x)$, $i = \overline{1,n}$ projection of the vector-function $H_r(x,t)$ to the subspace X_{ri}.

Then we have the following

Theorem 1. *For equation* (1) *let*

a) *the conditions* 1° *and* 2° *hold,*
b) *right side of the equation* (23) *belongs to SNS* (10) *and contains no element of the kernel of the operator* R_0*,*
c) $S_{r1}(0) = 0$.

Then there exists a solution of (23) *in the space* Y_r *and it can be represented in the form*

$$W_r(x,t) = Z_r(x,t) + y_r(x,t)\,. \tag{25}$$

Here

$$Z_r(x,t) = b_1(x)\Big[\sum_{k=1}^{2}[V_{r1k}(x)U_k(t_1) + Q_{r1k}(x)U_k'(t_1)] + \\ + f_{r1}(x)\nu(t_1) + g_{r1}(x)\nu'(t_1)\Big] + \sum_{k=1}^{2}\sum_{j=2}^{n} b_j(x)\alpha_{rjjk}(x)\exp(t_{jk})\,, \tag{26}$$

where the coefficients of ESM are arbitrary of the sufficiently smooth functions with $x \in I$ *and* $y_r(x,t)$ *uniquely defined and sufficiently smooth functions for all* $x \in I$*, in particular for* $x = 0$.

Remark 2. Since the point $x = 0$ is unstable, then ESM $U_2(t_1) \equiv Bi(t_1)$ and its derivative is unboundedly increasing when $t_1 \to +\infty$. However in spite of that in ESM the $\nu(t)$ and $\nu'(t)$ contain unboundedly increasing functions $Bi(t_1)$ and $B'i(t_1)$, they all are still bounded functions for $t \geq 0$ i.e. for $t \in [0, \mu^{-2}\varphi_1(1)]$.

Because of the shortness of the article we are not able to construct the full solution of at least three equations (20)–(22). We are bound to give the only remark.

Remark 3. Coefficients $\alpha_{rjjk}(x)$, $j = \overline{2, p-1}$ derived from the simple elements $\lambda_j(x)$ can be defined by scalar linear differential equations of the first order. Multiplicity of the element $\lambda_p(x) \equiv \cdots \equiv \lambda_n(x)$ introduces in the construction of asymptotic solutions of the extended problem (6) the following changes. Coefficients $\alpha_{rjjk}(x)$ $(j = \overline{p,n})$ derived from multiple elements of the spectrum can be determined by the system of $(n-p+1)$ differential equations.

6 Asymptotic correctness of iterated problems

Applying Theorem 1 consequently we can obtain solutions of the iterated equations (20)–(22) and so on. Each of those solutions contains $2n$ arbitrary constants which were obtained by integrations of differential equations and systems of differential equations with respect to unknown functions $V_{rlk}(x)$, $\alpha_{rjjk}(x)$, $j = \overline{2,n}$; $k = 1, 2$.

Substituting the obtained solutions $W_r(x,t)$ to the adequate boundary conditions one can obtain the system of $2n$ algebraic equations from which one can find unknown constants:

$$\varDelta(\varepsilon) C_r = \varGamma_r \,, \tag{27}$$

where

$$C_r = (V^0_{0111}, V^0_{0112}, \alpha^0_{0221}, \ldots, \alpha^0_{0nn1}, \alpha^0_{0222}, \ldots, \alpha^0_{0nn2})$$

is an unknown vector and $\varGamma_r$ a given vector.

One can show that the asymptotic equality holds:

$$\varDelta(\varepsilon) = K Bi\big(t_1(l)\big)\big[1 + O\big(Bi^{-1}(t_1(l))\big)\big] \,, \tag{28}$$

where

$$K = 2^{-\frac{2}{3}} \varGamma^{-1}(2/3) \prod_{k=1}^{2} b_{11}((k-1)l) \times \\ \times \prod_{j=2}^{p} \varphi'_{jk}((k-1)l) \cdot \big|\widetilde{B}_1((k-1)l)\big| \cdot \big|\widetilde{B}_2((k-1)l)\big| \,, \tag{29}$$

$$\widetilde{B}_1(x) = (b_{ij}(x))^{p-1}_{i,j=2}, \qquad \widetilde{B}_2(x) = (b_{ij}(x))^{n}_{i,j=p},$$

where $b_{ij}(x)$ is ith coordinate of the vector $b_j(x)$.

Lemma 4. *Let*

$$\big|B_s((k-1)l)\big| \neq 0, \qquad k, s = 1, 2 \tag{30}$$

Then for sufficiently small values of the parameter $\varepsilon > 0$ the determinant of the matrix $\varDelta(\varepsilon)$ is distinct from zero.

Thus if the conditions (30) holds, then each of the systems (19), where $r \geq -2$, has a unique solution, i.e. each of the functions $W_r(x,t)$ is uniquely determined as a solution of the adequate iterated problem.

7 Estimation of the last part of the asymptotic of a solution

Let us write the formal solution of the extended problem (6) in the form:

$$\widetilde{W}(x,t,\mu) \equiv W_{\varepsilon q}(x,t,\mu) + \mu^{q+1}\widetilde{\xi}_{q+1}(x,t,\varepsilon)\,, \tag{31}$$

where

$$W_{\varepsilon q}(x,t,\mu) \equiv \sum_{r=-2}^{q} \mu^r W_r(x,t) \text{ — } q\text{-partial sum of the series (7).}$$

If in the equality (31) we realized the narrowing by $t = \phi(x,\varepsilon)$ then we obtain the equality

$$W(x,\varepsilon) \equiv W(x,\phi(x,\varepsilon),\varepsilon) \equiv W_{\varepsilon q}(x,\phi,\varepsilon) + \varepsilon^{\frac{q+1}{3}}\xi_{q+1}(x,\phi,\varepsilon)\,. \tag{32}$$

Lemma 5. *If the conditions* 1° *and* 2° *hold then for sufficiently small values of the parameter* $\varepsilon > 0$ *:*

a) *the series* (19) *is an asymptotic series for the solution of the extended problem* (6)

b) *the narrowing of the series* (19) *by* $t = \phi(x,\varepsilon)$*, i.e. the series* (32)*, is asymptotic series for a solution SSPDE* (1)*.*

Applying Lemma 5 one can prove that the following asymptotic equality holds:

$$\xi_{q+1}(x,\phi,\varepsilon) \cong O(\mu^{-\frac{1}{2}}\exp\{\mu^{-3}(2/3)\varphi^{\frac{3}{2}}(x)\})\,. \tag{33}$$

The results obtained in the article can be formulated in the form of the following theorem:

Theorem 6. *If the conditions* 1° *and* 2° *hold then for sufficiently small values of the parameter* $\varepsilon > 0$*:*

a) *one can construct (applying the above described methods) a unique asymptotic series* (19) *as a solution of the extended problem* (6) *in SNS;*

b) *the narrowing of the series* (19) *for* $t = \phi(x,\varepsilon)$ (32) *is asymptotic series for a solution SSPDE* (1)*;*

c) *the last part of the asymptotic series of the solution* (1) *has the estimation* (33)*.*

Remark 7. Let $(h(0), b_1^*(0)) = 0$. Then

1) a solution of the degenerate equation (3) is a sufficiently smooth function for each $x \in [0,1]$;
2) a solution SSPDE (1) contains no negative degrees of a small parameter $\mu > 0$;
3) $\alpha_m = 0$ $(m = 0,1)$ in the boundary conditions for SSPDE (1), i.e. they represent the form like in the problems with stable spectrum of degenerate operator.

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