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A Functional Differential Equation in Banach Spaces

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Abstract. In this paper we prove the existence of pseudo-solution and weak solution for the Cauchy problem $x' = Fx$, $x(0) = x_0$, $t \in [0, a]$.

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The study of the Cauchy problem for differential and functional differential equations in a Banach space relative to the strong topology has attracted much attention in recent years. However a similar study relative to the weak topology was studied by many authors, for example, Szep [11], Mitchell and Smith [9], Szufla [12], Kubiaczyk [6,7], Kubiaczyk and Szufla [8], Cichoń [1], Cichoń and Kubiaczyk [2], and others.

Let E be a Banach space, E^* the dual space. We set $B_b(x_0) = \{x \in E : \|x - x_o\| \leq b\}$, $(b > 0)$. We denote by $C(I, E)$ the space of all continuous function from I to E, and by $(C(I, E), w)$ the space $C(I, E)$ with the weak topology. Put

$$\widetilde{B} = \{x \in C(J, E) : x(J) \subset B_b(x_o), \|x(t) - x(s)\| \leq M|t - s|, \text{ for } t, s \in J\} \ ,$$

note that $\widetilde{B}$ is nonempty, closed, bounded, convex and equicontinuous, where $J = [0, h]$, $h = \min\{a, \frac{b}{M}\}$ and $M > 0$ is a constant.

We deal with the Cauchy problem:

$$x' = Fx, \quad x(0) = x_0, \quad t \in I = [0, a], \tag{1}$$

in the case of F being an bounded operator of Volterra type from $\widetilde{B}$ into $P(I, E)$ (the space of all Pettis integrable functions on I).

Let us introduce the following definitions.

This is the final form of the paper.

Definition 1. F is said to be of Volterra type if for $x_1, x_2 \in \widetilde{B}$ and for any $s_o > 0$ the equality $x_1(t) = x_2(t)$ for $t < s_o$ implies $(Fx_1)(t) = (Fx_2)(t)$ for $t \leq s_o$.

Now fix $x^* \in E^*$, and consider

$$(x^*x)'(t) = x^*((Fx)(t)), \quad t \in I. \tag{1'}$$

Definition 2. A function $x : I \longrightarrow E$ is said to be a pseudo-solution of the Cauchy problem (1) if it satisfies the following conditions:

(i) $x(\cdot)$ is absolutely continuous,

(ii) $x(0) = x_o$,

(iii) for each $x^* \in E^*$ there exists a negligible set $A(x^*)$ (i.e., mes $(A(x^*)) = 0$), such that for each $t \notin A(x^*)$,

$$x^*(x'(t)) = x^*((Fx)(t)) \ .$$

Here $'$ denotes a pseudoderivative (see Pettis [10]).

In other words, by a pseudo-solution of (1) we will mean an absolutely continuous function $x(\cdot)$, with $x(0) = x_o$, satisfying (1′) a.e. for each $x^* \in E^*$.

Definition 3. A function $r : [0, \infty) \longrightarrow [0, \infty)$ is said to be a Kamke function if it satisfies the following conditions:

(i) $r(0) = 0$,

(ii) $u(t) \equiv 0$ is the unique solution of the integral equation

$$z(t) = \int_0^t r(z(s))ds \quad , \quad t \in I \ .$$

Lemma 4 ([9]). *Let $H \subset C(I, E)$ be a family of strongly equicontinuous functions. Then*

$$\beta_c(H) = \sup_{t \in I} \beta(H(t)) = \beta(H(I)) \ ,$$

where $\beta_c(H)$ denote the measure of weak noncompactness in $C(I, E)$ and the function $t \rightarrow \beta(H(t))$ is continuous.

Now suppose that:

($*$) For each strongly absolutely continuous function $x : J :\longrightarrow E$, $(Fx)(\cdot)$ is Pettis integrable, $F(\cdot)$ is weakly-weakly sequentially continuous, then the existence of a pseudo-solution of (1) is equivalent to the existence of a solution for

$$x(t) = x_o + \int_0^t (Fx)(s)ds \ , \tag{2}$$

where the integral is in the sense of Pettis (see [10]).

Theorem 5. *Let F be a bounded continuous operator of Volterra type from $\widetilde{B}$ into $P(I,E)$ and under the assumption (*) and*

$$\beta\Big(\bigcup\{(Fx)[J] : x \in \widetilde{X}\}\Big) \leq r(\beta(\widetilde{X})) \ , \tag{3}$$

holds for every subset $\widetilde{X}$ of $\widetilde{B}$, where r is a non-decreasing Kamke function and β is the measure of weak noncompactness. Then the set S of all pseudo-solutions of the Cauchy problem (1) on J is non-empty and compact in $(C(J,E),w)$.

Proof. Put

$$Tu(t) = x_o + \int_0^t Fu(s)ds \quad , \quad t \in I, \quad u \in \widetilde{B} \ ,$$

where the integral is in the sense of Pettis.

By our assumptions the operator T is well defined and maps $\widetilde{B}$ into $\widetilde{B}$.

Using Lebesgue's dominated convergence theorem for the Pettis integral (see [4]), we deduce that T is weakly sequentially continuous.

Suppose that $\overline{V} = \overline{\text{Conv}}(\{x\} \cup T(V))$ for some $V \subset \widetilde{B}$. We will prove that V is relatively weakly compact, thus Theorem 1 in [7] is satisfied.

From the definition of $\widetilde{B}$ and Lemma 4 it follows that the function $v : t \to \beta(V(t))$ is continuous on J.

For fixed $t \in J$, divide the interval $[0,t)$ into m parts:

$$0 = t_o < t_1 < \cdots < t_m = t, \text{ where } \quad t_i = it/m \quad , \quad i = 0,1,2,\ldots,m \ .$$

Put

$$V([t_{i-1},t_i]) = \{u(s) = u \in V, \quad t_{i-1} \leq s \leq t_i\} \ .$$

By Lemma 4 and the continuity of v there is $s_i \in [t_{i-1},t_i]$ such that

$$\beta(V([t_{i-1},t_i])) = \sup\{\beta(V(s)) : t_{i-1} \leq s \leq t_i\} = v(s_i) \ . \tag{4}$$

On the other hand, by the mean value theorem we obtain

$$Tu(t) = x_o + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} Fu(s)ds \in x_o + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\text{Conv}}Fu([t_i,t_{i+1}])$$

for each $u \in V$. Therefore

$$TV(t) \subset x_o + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\text{Conv}}F([V])([t_i,t_{i+1}]) \, .$$

By (4) and the corresponding properties of β it follows that

$$\begin{aligned}\beta(T(V)(t)) \leq \beta(x_o + \sum_{i=0}^{m-1}(t_{i+1}-t_i)\overline{\text{Conv}}F([V])([t_i,t_{i+1}])) &\leq \\ \leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\beta(F(V)([t_i,t_{i+1}])) &\leq \\ \leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)r(\beta(V[t_i,t_{i+1}])) &\leq \\ \leq \sum_{i+0}^{m-1}(t_{i+1}-t_i)r(\beta(V(s_i)) \ , \text{ for some } s_i \in [t_i,t_{i+1}] \\ = \sum_{i=0}^{m-1}(t_{i+1}-t_i)r(v(s_i)) \ .\end{aligned}$$

By letting $m \to \infty$, we have

$$\beta(T(V(t)) \leq \int_0^t r(v(s))ds \ . \tag{5}$$

Since $\overline{V} = \overline{\text{Conv}}(\{x\} \cup T(V))$ we have $\beta(V(t)) \leq \beta(T(V(t)))$ and in view of (5), it follows that $v(t) \leq \int_0^t r(v(s))ds$ for $t \in J$.

Hence applying now a theorem on differential inequalities (cf. [5]) we get $v(t) = \beta(v(t)) = 0$.

By Lemma 4, V is relatively weakly compact.

So, by Theorem 1 in [7], T has a fixed point in $\widetilde{B}$ which is actually a pseudo-solution of (1).

As $S = T(S)$, by repeating the above argument with $V = S$ we can show that S is relatively compact in $(C(J,E),w)$.

Since T is weakly sequentially continuous on $\overline{S(J)}^{\omega}$, S is weakly sequentially closed. By Eberlein-Smulian Theorem [3], S is weakly compact.

Remark 6. One can easily prove that the integral of a weakly continuous function is weakly differentiable with respect to the right endpoint of the integration interval and its derivative equals the integral at the same point (see [6], Lemma 2.3). In this case a pseudo-solution is, actually, a weak solution. Moreover, in some classes of spaces our pseudo-solutions are also strong C-solutions (in separable Banach spaces, for instance).

References

[1] Cichoń, M., *Weak solutions of differential equations in Banach spaces.* Discuss. Math. — Diff. Inclus. **15** (1995), 5–14.

[2] Cichoń, M. and Kubiaczyk, I., *On the set of solutions of the Cauchy problem in Banach spaces*. Arch. Math. **63** (1994), 251–257.

[3] Edwards, R. E., *Functional Analysis*. Holt Rinehart and Winston, New York 1965.

[4] Geitz, R. F., *Pettis integration* . Proc. Amer. Math. Soc. **82** (1981), 81–86.

[5] Hartman, P., *Ordinary Differential Equations*. New York 1964.

[6] Kubiaczyk, I., *A functional differential equation in Banach spaces*. Demonstratio Math. **15** (1982), 113–129.

[7] Kubiaczyk, I., *On a fixed point theorem for weakly sequentially continuous mappings*. Discuss. Math. — Diff. Inclus. **15** (1995), 15–20.

[8] Kubiaczyk, I. and Szufla, S., *Kneser's theorem for weak solutions of ordinary differential equations in Banach spaces*. Publ. Inst. Math. **32** (1982), 99–103.

[9] Mitchell, A. R. and Smith, C., *An existence theorem for weak solutions of differential equations in Banach spaces*. pp. 387–404 in, Nonlinear Equations in Abstract Spaces, ed. by V. Lakshmikantham 1978.

[10] Pettis, B. J., *On integration in vector spaces*. Trans. Amer. Math. Soc. **44** (1938), 277–304.

[11] Szep, A., *Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces*. Studia Sci. Math. Hungar. **6** (1971), 197–203.

[12] Szufla, S., *Kneser's theorem for weak solutions of ordinary differential equations in reflexive Banach spaces*. Ibid. **26** (1978), 407–413.