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A Time Periodic Solution of the Navier-Stokes Equations with Mixed Boundary Conditions

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Abstract. We study qualitative properties of the system of time-periodic Navier-Stokes equations and continuity equation with the Dirichlet boundary condition on the fixed wall and the natural boundary condition on the input and on the output.

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1 Description of the Domain

We suppose that $\Omega \subset \mathbb{R}^n$, where $n = 2$ or $n = 3$, Ω is a bounded domain, $\partial\, \Omega \in \mathbb{C}^{0,1}$. Further, we suppose that $\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are closed (not necessarily connected) sets such that $\mathrm{meas}_{n-1}(\Gamma_1 \cap \Gamma_2) = 0$ and $\mathrm{meas}_{n-1}(\Gamma_1) > 0$.

The domain Ω corresponds to a channel filled up by a fluid. Γ_1 is a fixed wall of the channel and Γ_2 involves the input and the output of the channel.

2 Classical Formulation of the Problem

Let $T > 0$ be a positive number. $(0, T)$ denotes the time interval, $Q = \Omega \times (0, T)$, $e_{ij}(u)$ (for $1 \le i, j \le n$) denotes $\partial u_i/\partial x_j + \partial u_j/\partial x_i$.

The problem we will deal with can be classically formulated as follows:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_j}(\nu \cdot e_{ij}(u)) + u_j \cdot \frac{\partial u_i}{\partial x_j} + \frac{\partial \mathcal{P}}{\partial x_i} = g_i \quad \text{in } Q, \quad i = 1, \ldots, n, \tag{1}$$

$$\mathrm{div} u \;= 0 \quad \text{in } Q, \tag{2}$$

$$u = 0 \quad \text{in } \Gamma_1 \times (0, T), \tag{3}$$

$$-\mathcal{P} \cdot \mathbf{n}_i + \nu \cdot e_{ij}(u) \cdot \mathbf{n}_j = \sigma_i \quad \text{in } \Gamma_2 \times (0, T), \tag{4}$$

This is the final form of the paper.

$$u(x,0) = u(x,T) \text{ in } \Omega, \tag{5}$$

$$u(.,0) = 0 \text{ on } \Gamma_1. \tag{6}$$

Here u is the velocity, $\mathcal{P}$ is the pressure, ν denotes the viscosity, g is a body force, σ is a prescribed vector function on Γ_2 and $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_n)$ is the outer normal vector. The problem (1)–(6) will be called the time-periodic Navier-Stokes problem with the mixed boundary conditions. We suppose that ν is a positive constant in the whole paper.

The used Dirichlet boundary condition expresses a non-slip behaviour of the fluid on the fixed walls of the channel. The condition (4) means that we prescribe a normal component of the stress tensor on Γ_2. The Navier-Stokes equations with condition (4) were already treated in the works [1]–[6].

3 Some Function Spaces and Their Properties

To formulate the problem (1)–(6) weakly, we shall need some function spaces. Let us denote

$$\mathcal{E}(\overline{\Omega}) = \{\varphi \in [\mathcal{C}^\infty(\overline{\Omega})]^n;\ \text{div}\ \varphi \equiv 0,\ \text{supp}\ \varphi \cap \Gamma_1 \equiv \emptyset\}.$$

The Banach spaces $V^{k,p}$, resp. $V^{0,q}$, is defined as the closure of $\mathcal{E}(\overline{\Omega})$ in the norm of the space $[W^{k,p}(\Omega)]^n$, resp. $[L^p(\Omega)]^n$, where $k > 0$ (it need not be an integer) and $1 \leq q \leq \infty$. For simplicity, we denote the space $V^{0,2}$ by the symbol H.

Both the spaces $V^{1,2}$ and H are Hilbert spaces with the scalar products

$$((\psi, \phi))_{1,2} = \int_\Omega e_{ij}(\psi) \cdot e_{ij}(\phi)\ d(\Omega)$$

resp.

$$((\psi, \phi))_{0,2} = \int_\Omega \psi_i \cdot \phi_i\ d(\Omega).$$

The symbol $\langle ., . \rangle$ denotes the duality between elements from $(V^{1,2})^*$ and $V^{1,2}$.

It is obvious that $V^{1,2}$, H and $(V^{1,2})^*$ are three Hilbert spaces, which satisfy the following conditions

$$V^{1,2} \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow (V^{1,2})^*$$

and H coincides with the interpolation $[V^{1,2}, (V^{1,2})^*]_{1/2}$. Moreover, if $u \in L^2(0,T,V^{1,2})$, $u' \in L^2(0,T,(V^{1,2})^*)$, then $u \in \mathcal{C}([0,T];H)$ and

$$\|u\|_{L^\infty(0,T;H)} \leq c \cdot (\|u\|_{L^2(0,T;V^{1,2})} + \|u'\|_{L^2(0,T;(V^{1,2})^*)}),$$

where $c = c(\Omega)$.

If $\mathcal{X}$ is a Banach space then $(\mathcal{X})^*$ will denote its dual and $L^p(0,T;\mathcal{X})$, $1 < p < \infty$, will be the linear space of all measurable functions from the interval $(0,T)$ into $\mathcal{X}$ such that

$$\int_0^T \|u(t)\|^p_{\mathcal{X}}\ dt < \infty.$$

Let X and Y be the following Banach spaces:

$$X = \{u; u' \in L^2(0,T,V^{1,2}), u'' \in L^2(0,T,(V^{1,2})^*), u(0) = u(T) \in V^{1,2}, \\ u'(0) = u'(T) \in H\},$$

$$\|u\|_X = \|u\|_{L^2(0,T;V^{1,2})} + \|u'\|_{L^2(0,T;V^{1,2})} + \|u''\|_{L^2(0,T;(V^{1,2})^*)},$$

$$Y = \{f; f \in \mathcal{C}([0,T],(V^{1,2})^*), f' \in L^2(0,T,(V^{1,2})^*), f(0) = f(T) \in (V^{1,2})^*\},$$

$$\|f\|_Y = \|f\|_{L^2(0,T;(V^{1,2})^*)} + \|f'\|_{L^2(0,T;(V^{1,2})^*)}.$$

4 Weak Formulation of the Problem

The weak formulation of the problem (1)–(6) will be based on an operator equation. Therefore we define operators $\mathcal{S}$, $\mathcal{B}$ and $\mathcal{N}$ at first.

The operator $\mathcal{S}$ from X to Y is defined by the equation

$$\langle \mathcal{S}(u), v\rangle = (\!(u', v)\!)_{0,2} + \nu \cdot (\!(u, v)\!)_{1,2}$$

for every $v \in V^{1,2}$ and almost every $t \in (0,T)$.

$b(\varphi, \psi, \phi)$ will denote trilinear form on $V^{1,2} \times V^{1,2} \times V^{1,2}$ such that

$$b(\varphi, \psi, \phi) = \int_\Omega \varphi_j \cdot \frac{\partial \psi_i}{\partial x_j} \cdot \phi_i \; d(\Omega).$$

It can be easily verified that $b(\varphi, \psi, \phi)$ satisfies the following estimate

$$|b(\varphi, \psi, \phi)| \le c \cdot \|\varphi\|_{V^{1,2}} \cdot \|\phi\|_{V^{1,2}} \cdot \|\psi\|_{V^{1,2}}, \tag{7}$$

where $c = c(\Omega)$.

Integrating by parts and using the theorems about imbedding the space $[W^{k_p}(\Omega)]^n$ into the space $L^q(\partial\Omega)$ the following estimates are verified:

$$|b(\varphi, \psi, \phi)| \le c \cdot \|\varphi\|_{V^{1,2}} \cdot \|\psi\|_{V^{\frac{7}{8},2}} \cdot \|v\|_{V^{1,2}}, \tag{8}$$

$$|b(\varphi, \psi, \phi)| \le c \cdot \|\varphi\|_{V^{\frac{7}{8},2}} \cdot \|\psi\|_{V^{1,2}} \cdot \|v\|_{V^{1,2}}. \tag{9}$$

The symbols φ and ψ will sometimes also denote functions of the variable t with values in $V^{1,2}$.

$\mathcal{B}$ will be operator from X into Y defined by the equation

$$\langle \mathcal{B}(u), v\rangle = b(u, u, v)$$

for every $v \in V^{1,2}$ and almost every $t \in (0,T)$.

Finally, operator $\mathcal{N}$ from X into Y is defined by the equation

$$\mathcal{N}(u) = \mathcal{S}(u) + \mathcal{B}(u).$$

A function $u \in X$ will be called a weak solution to the time-periodic Navier-Stokes problem with the right hand side f if

$$\mathcal{N}(u) = f.$$

Notice that

$$\langle f, v \rangle = \int_{\Omega} g_i \cdot v_i \; d(\Omega) + \int_{\Gamma_2} \sigma_i \cdot v_i \; d(\partial\Omega).$$

5 The Local Diffeomorphism Theorem

Suppose that u_0 and f_0 are such elements of X and Y that

$$\mathcal{N}(u_0) = f_0.$$

(This means that u_0 is a weak solution to the time-periodic Navier-Stokes problem with the right hand side f_0.) Our further aim is to investigate the solvability of the equation $\mathcal{N}(u) = f$ with f from some neighbourhood of point f_0 in Y. To solve this problem, we will use the following very important theorem (the Local Diffeomorphism Theorem).

Theorem 1. *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, $\mathcal{F}$ be a mapping from $\mathcal{X}$ into $\mathcal{Y}$ belonging to C^1 in some neighbourhood V of point u_0. If $\mathcal{F}'(u_0)$ is one-to-one from $\mathcal{X}$ onto $\mathcal{Y}$ and continuous, then there exists a neighbourhood U of point u_0, $U \subset V$ and a neighbourhood W of point $f(u_0)$, $W \subset \mathcal{Y}$, so that $\mathcal{F}$ is one-to-one from U to W.*

6 The Fréchet Derivative of Operator $\mathcal{N}$

It is obvious that if there exists a point $u \in X$ in which the operator $\mathcal{N}$ satisfies the assumption of the Local Diffeomorphism Theorem then the equation $\mathcal{N}(u) = f$ is "locally solvable" (i.e. solvable in some neighbourhood of u). It is clean that $\mathcal{N} \in \mathcal{C}^1(X)$. Further, need to express the Fréchet derivative of operator $\mathcal{N}$ at point u and to find out, whether it is one-to-one. We will express the Fréchet derivative of $\mathcal{N}$ by means of operators $\mathcal{K}$ and $\mathcal{G}$.

$\mathcal{K}$ is the bilinear operator from $X \times X$ into Y defined by the equation

$$\langle \mathcal{K}(u), v \rangle = b(u, w, v) + b(w, u, v)$$

for every $v \in V^{1,2}$ and for almost every $t \in (0, T)$.

The operator $\mathcal{G}$ from $X \times X$ into Y is given by the equation

$$\mathcal{G}(u,w) = \mathcal{S}(w) + \mathcal{K}(u,w).$$

It is possible to prove there exists a constant $c = c(\Omega)$ so that

$$\|b(u,w,.)\|_Y \leq c \cdot \|u\|_X \cdot \|w\|_X.$$

Theorem 2. *Let $u \in X$. Then the operator $\mathcal{G}(u,.)$ is the Fréchet derivative of $\mathcal{N}$ at point u and $\mathcal{G} \in \mathcal{C}^1(X \times X, Y)$.*

Proof. It is possible to prove for arbitrary $u, w \in X$ following estimate

$$\|b(u,w,.)\|_Y \leq c \cdot \|u\|_X \cdot \|w\|_X,$$

where $c = c(\Omega)$. Therefore and from the estimate

$$\|\mathcal{N}(u+w) - \mathcal{N}(u) - \mathcal{G}(u,w)\|_Y = \|b(w,w,.)\|_Y \leq c \cdot \|w\|_X^2,$$

we get

$$\lim_{\|w\|_X \to 0} \frac{\|\mathcal{N}(u+w) - \mathcal{N}(u) - \mathcal{G}(u,w)\|_Y}{\|w\|_X} = 0.$$

So $\mathcal{G}(u,.)$ is the Fréchet derivative of $\mathcal{N}$ at point u. The smoothness of $\mathcal{G}$ follows immediately from its definition. The proof is complete.

7 Local Properties of Operator $\mathcal{N}$

We have proved that the operator $\mathcal{G}(u,.)$ has the form

$$\mathcal{G}(u,.) = \mathcal{S}(.) + \mathcal{K}(u,.)$$

in the previous section. Further we will prove that operator $\mathcal{S}$ is a one-to-one linear operator from X onto X and $\mathcal{K}(u,.)$ is a compact linear operator from X into Y. So the operator $\mathcal{G}(u,.)$ is the sum of a one-to-one operator and a compact operator. Operators of this form have properties which will be used later.

Lemma 3. *$\mathcal{S}$ is a linear continuous one-to-one operator from X onto Y.*

Proof. The linearity and continuity of $\mathcal{S}$ are obvious. Next we prove that $\mathcal{S}$ is an operator from X onto Y. The form $(\!(.,.)\!)_{1,2}$ is $V^{1,2}$-elliptic. Then there exists $w \in L^2(0,T,V^{1,2}) \cap \mathcal{C}([0,T];H)$, so that $w' \in L^2(0,T,(V^{1,2})^*)$, the equation

$$\frac{d}{dt}(\!(w(t),v)\!)_{0,2} + \nu \cdot (\!(w(t),v)\!)_{1,2} = \langle f',v\rangle$$

holds for every $v \in V^{1,2}$ and

$$w(0) = w(T).$$

Then there exists $\omega_0 \in V^{1,2}$ so that for every $v \in V^{1,2}$ holds

$$\nu \cdot ((\omega_0, v))_{1,2} = \langle f, v\rangle - ((w(0), v))_{0,2}.$$

Let

$$u(t) = \omega_0 + \int_0^t w(s)\ ds,\ \ t \in (0, T).$$

Then $u \in X$ and $\mathcal{S}(u) = f$. Thus we have proved that $\mathcal{S}$ is from X onto Y. Let us suppose that $\mathcal{S}(u) = 0$. Then $u = 0$. The proof is complete.

Lemma 4. *Let $u \in X$. Then $\mathcal{K}(u, .)$ is a linear compact operator from X into Y.*

Prior to the proof we recall a result from [7, Lemma 4.5]. Denote

$$\mathcal{Z} = \{u; u \in L^2(0, T, V^{1,2}), u' \in L^2(0, T, (V^{1,2})^*)\}$$

with the norm

$$\|u\|_{\mathcal{Z}} = \|u\|_{L^2(0,T;V^{1,2})} + \|u'\|_{L^2(0,T;(V^{1,2})^*)}$$

(u' is the Schwartz derivative in the sence of imbedding $V^{1,2} \hookrightarrow H \hookrightarrow (V^{1,2})^*$). Then

$$\mathcal{Z} \hookrightarrow\hookrightarrow L^2(0, T; V^{\frac{7}{8},2}) \tag{10}$$

Proof. Let $w_k \subset X$ be a bounded set in X. Using (10) we get $w \in X$ such that

$$w_k' \to w' \text{ in } L^2(0, T; V^{\frac{7}{8},2}) \tag{11}$$

and

$$w_k(0) \to w(0) \text{ in } V^{\frac{7}{8},2}$$

Combining it with (11) we get

$$w_k \to w \text{ in } L^\infty(0, T; V^{\frac{7}{8},2}). \tag{12}$$

Note that

$$\begin{aligned}
&\|\mathcal{K}(u, w_k) - \mathcal{K}(u, w)\|_Y = \\
&\quad = \|b(u, w_k - w, .)\|_{L^2(0,T;(V^{1,2})^*)} + \|b(w_k - w, u, .)\|_{L^2(0,T;(V^{1,2})^*)} + \\
&\quad + \|b(u, w_k' - w', .)\|_{L^2(0,T;(V^{1,2})^*)} + \|b(u', w_k - w, .)\|_{L^2(0,T;(V^{1,2})^*)} + \\
&\quad + \|b(w_k' - w', u, .)\|_{L^2(0,T;(V^{1,2})^*)} + \|b(w_k - w, u', .)\|_{L^2(0,T;(V^{1,2})^*)}
\end{aligned} \tag{13}$$

We estimate the third and fourth additive terms. Let $v \in V^{1,2}$. Use (8) to get the estimate

$$|b(u(t), w_k'(t) - w'(t), v)| \le c \cdot \|u(t)\|_{V^{1,2}} \cdot \|w_k'(t) - w'(t)\|_{V^{\frac{7}{8},2}} \cdot \|v\|_{V^{1,2}}.$$

Therefore

$$\|b(u(t), w_k'(t) - w'(t), .)\|_{(V^{1,2})^*} \leq c \cdot \|u(t)\|_{V^{1,2}} \cdot \|w_k(t) - w(t)\|_{V^{\frac{7}{8},2}}$$

for almost all $t \in (0,T)$, $c = c(\Omega)$. It follows that

$$\|b(u, w_k' - w', .)\|_{L^2(0,T;(V^{1,2})^*)} \leq c \cdot \|u\|_{L^\infty(0,T;V^{1,2})} \cdot \|w_k' - w'\|_{L^2(0,T;V^{\frac{7}{8},2})}. \quad (14)$$

Similarly, we get

$$|b(u'(t), w_k(t) - w(t), v)| \leq c \cdot \|u'(t)\|_{V^{1,2}} \cdot \|w_k(t) - w(t)\|_{V^{\frac{7}{8},2}} \cdot \|v\|_{V^{1,2}}$$

and therefore

$$\|b(u'(t), w_k(t) - w(t), .)\|_{(V^{1,2})^*} \leq c \cdot \|u'(t)\|_{V^{1,2}} \cdot \|w_k(t) - w(t)\|_{V^{\frac{7}{8},2}}$$

for almost all $t \in (0,T)$, $c = c(\Omega)$. It follows

$$\|b(u', w_k - w, .)\|_{L^2(0,T;(V^{1,2})^*)} \leq c \cdot \|u'\|_{L^2(0,T;V^{1,2})} \cdot \|w_k - w\|_{L^\infty(0,T;V^{\frac{7}{8},2})}. \quad (15)$$

The same way we prove

$$\|b(u, w_k - w, .)\|_{L^2(0,T;(V^{1,2})^*)} \leq c \cdot \|u\|_{L^2(0,T;(V^{1,2})^*)} \cdot \|w_k - w\|_{L^\infty(0,T;V^{\frac{7}{8},2})}, \quad (16)$$

$$\|b(w_k - w, u, .)\|_{L^2(0,T;(V^{1,2})^*)} \leq c \cdot \|u\|_{L^2(0,T;(V^{1,2})^*)} \cdot \|w_k - w\|_{L^\infty(0,T;V^{\frac{7}{8},2})}, \quad (17)$$

$$\|b(w_k' - w', u, .)\|_{L^2(0,T;(V^{1,2})^*)} \leq c \cdot \|u\|_{L^\infty(0,T;V^{1,2})} \cdot \|w_k' - w'\|_{L^2(0,T;V^{\frac{7}{8},2})} \quad (18)$$

and

$$\|b(w_k - w, u', .)\|_{L^2(0,T;(V^{1,2})^*)} \leq c \cdot \|u'\|_{L^2(0,T;(V^{1,2})^*)} \cdot \|w_k - w\|_{L^\infty(0,T;V^{\frac{7}{8},2})}. \quad (19)$$

From (11)–(19) we get

$$\|\mathcal{K}(u, w_k) - \mathcal{K}(u, w)\|_Y \to 0.$$

The proof is complete.

The operator $\mathcal{G}(u, .)$ is the sum of a one-to-one operator and a compact operator. The operators of this form are widely treated in mathematical literature and we can apply their known properties to prove the following theorem.

Theorem 5. *Let $u \in V^{1,2}$. Then the following statements are equivalent:*

(a) $\mathcal{G}(u, .)$ *is an injective operator .*

(b) $\mathcal{G}(u, .)$ *is an operator onto* $(V^{1,2})^*$.

Moreover, if the statements (a)–(b) *are satisfied at point u then there exists an open neighbourhood U of point u in X and an open neighbourhood W of point $\mathcal{N}(u)$ in Y such that $\mathcal{N}$ is a one-to-one operator from U onto W.*

References

[1] R. Glowinski, *Numerical methods for nonlinear variational problems*, Springer Verlag, Berlin-Heidelberg-Tokio-New York, 1984.

[2] S. Kračmar, J. Neustupa, Global existence of weak solutions of a nonsteady variational inequalities of the Navier-Stokes type with mixed boundary conditions. *Proc. of the conference ISNA'92*, (1992), Part III, 156–157.

[3] S. Kračmar, J. Neustupa, Modelling of flows of a viscous incompressible fluid through a channel by means of variational inequalities. *ZAMM 74*, **6** (1994), 637–639.

[4] S. Kračmar, J. Neustupa, Some Initial Boundary Value Problems of the Navier-Stokes Type with Mixed Boundary Conditions. *Proc. of the seminar Numerical Mathematics in Theory and Practice*, Pilsen, (1993), 114–120.

[5] S. Kračmar, J. Neustupa, Simulation of Steady Flows through Channels by Variational Inequalities. *Proc. of the conference Numerical Modelling in Continuum Mechanics, Prague 1994*, (1995), 171–174.

[6] R. Rannacher, Numerical analysis of the Navier-Stokes equations. *Proceedings of conference ISNA'92, part I.* (1992), 361–380.

[7] P. Kučera, Z. Skalák, Local solutions to the Navier-Stokes equations with mixed boundary conditions. *Submitted to Acta Applicandae Mathematicae.*