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Branching of Periodic Orbits in Hamiltonian and Reversible Systems

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Abstract. In this paper we survey a number of results on periodic orbits in Hamiltonian and reversible systems: the appearance of such orbits in one-parameter families, the bifurcation of such families from equilibria, the period blow-up near homoclinics, the branching of subharmonics and the phenomenon of subharmonic cascades.

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1 Introduction

One of the characteristic properties of Hamiltonian and reversible systems is that (symmetric) periodic orbits of such systems typically appear in one-parameter families, in contrast to periodic orbits of general systems which are typically limit cycles, i.e. they are isolated. Starting from this observation one can raise a number of questions, such as (1) how do branches of periodic orbits originate or terminate? (2) is there any "branching", i.e. can one branch of periodic orbits bifurcate from another such branch? and (3) how does this branching process change when parameters in the system are changed? In this paper we survey a number of results on these issues which we obtained in recent years in collaboration with Bernold Fiedler, Jan-Cees van der Meer, Jürgen Knobloch and Maria-Cristina Ciocci.

We will consider two different types of systems, namely Hamiltonian systems from one side, and reversible systems from the other side. Although in practice many Hamiltonian systems are also reversible, the two classes do not coincide, and we will treat them here strictly separated. Some of the results which we quote for Hamiltonian systems remain valid for the much larger class of conservative systems, i.e. for systems which have a first integral. Also, some of the results are for fixed systems, while others require one or more external parameters.

The Hamiltonian systems which we will consider have the form

$$\dot{x} = X_H(x,\lambda) := J\nabla_x H(x,\lambda), \tag{1.1}$$

where $x \in \mathbb{R}^{2n}$, $\lambda \in \mathbb{R}^m$, $H : \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}$ is a smooth function (the Hamiltonian), and $J \in \mathcal{L}(\mathbb{R}^{2n})$ is the standard symplectic matrix defined by $J(y,z) := (z,-y)$ for all $y,z \in \mathbb{R}^n$. It is immediate to see that $H(\cdot,\lambda)$ is a first integral for $(1.1)_\lambda$. We also consider reversible systems of the form

$$\dot{x} = f(x,\lambda), \tag{1.2}$$

again with $x \in \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}^m$, and with $f : \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}^{2n}$ a smooth parameter-dependent vectorfield such that

$$f(Rx,\lambda) = -Rf(x,\lambda) \tag{1.3}$$

for some linear operator $R \in \mathcal{L}(\mathbb{R}^{2n})$ satisfying $R^2 = I$ (i.e. R is a linear involution on $\mathbb{R}^{2n}$) and $\dim \mathrm{Fix}(R) = n$. If $\tilde{x}(t)$ is a solution of (1.2) then so is $\tilde{y}(t) := R\tilde{x}(-t)$; a (maximal) solution of (1.2) with orbit γ is called *symmetric* if $R\gamma = \gamma$.

We first show why periodic orbits of (1.1) or (1.2) appear typically in one-parameter families (at fixed values of the parameter λ). Let γ_0 be a periodic orbit of a Hamiltonian vectorfield X_H, let Σ be a transversal section to γ_0 at a point $x_0 \in \gamma_0$, and let $P : \Sigma \to \Sigma$ be the corresponding Poincaré mapping. For each $h \in \mathbb{R}$ near $h_0 := H(x_0)$ we set $\mathcal{E}_h := \{x \in \mathbb{R}^{2n} \mid H(x) = h\}$ and $\Sigma_h := \Sigma \cap \mathcal{E}_h$. Since H is a first integral for X_H it follows that P leaves each Σ_h invariant, which allows us to define $P_h : \Sigma_h \to \Sigma_h$ as the restriction of P to Σ_h. Clearly x_0 is a fixed point of P_{h_0}, and if 1 is not an eigenvalue of DP_{h_0} (which is typically the case) this fixed point persists for all nearby values of h. Hence we obtain a 1-parameter family of periodic orbits parametrized by the "energy" h. In the reversible case we use the property that a nontrivial orbit γ is symmetric and periodic if and only if γ intersects $\mathrm{Fix}\,(R)$ in exactly two points; the period then equals twice the time needed to travel along γ between these two points. Now suppose that γ_0 is a symmetric periodic orbit for a reversible vectorfield $f(x)$, with minimal period $T_0 > 0$, and let x_0 and y_0 be the two intersection points of γ_0 and $\mathrm{Fix}\,(R)$. Then x_0 and y_0 also belong to the intersection of $\mathrm{Fix}\,(R)$ with $\phi_{T_0/2}(\mathrm{Fix}\,(R))$ (where $\phi_t(x)$ denotes the flow of f), and generically this intersection will be transversal. If this is the case then the two intersection points will persist for nearby values of T, i.e. for each T near T_0 the intersection of $\mathrm{Fix}\,(R)$ with $\phi_{T/2}(\mathrm{Fix}\,(R))$ will contain two points x_T and y_T which generate a symmetric T-periodic orbit of f. We conclude that typically symmetric periodic orbits of reversible vectorfields appear in one-parameter families parametrized by the period T.

In the main part of this paper we will discuss how branches of (symmetric) periodic orbits can originate at equilibria (Section 2) or terminate at homoclinics (Section 3); we will also show how the bifurcation of subharmonic solutions leads to "branching" (Section 4). Finally we will very briefly discuss the phenomenon of subharmonic cascades (Section 5).

2 Branches originating at equilibria

The simplest conditions under which a branch of periodic orbits can originate from an equilibrium are given by the classical Liapunov Center Theorem. In the Hamiltonian case this theorem reads as follows.

Theorem 1. *Consider a Hamiltonian vectorfield X_H and let $x_0 \in \mathbb{R}^{2n}$ be such that:*

- (i) $X_H(x_0) = 0$;
- (ii) $A_0 := DX_H(x_0)$ *has a pair of simple purely imaginary eigenvalues* $\pm i\omega_0$ *(with* $\omega_0 > 0$*);*
- (iii) (nonresonance condition) A_0 *has no other eigenvalues of the form* $\pm ik\omega_0$*, with* $k \in \mathbb{Z}$*,* $k \neq \pm 1$*.*

Then the vectorfield X_H has a smooth 2-dimensional locally invariant manifold containing x_0 and foliated by periodic orbits surrounding x_0. As one moves along this 1-parameter family of periodic orbits towards x_0 the minimal period tends to $T_0 := 2\pi/\omega_0$. □

In the reversible case a similar result holds:

Theorem 2. *Let f be a reversible vectorfield, and let $x_0 \in \mathrm{Fix}\,(R)$ be a symmetric equilibrium of f such that the linearization $A_0 := Df(x_0)$ has a pair of simple purely imaginary eigenvalues $\pm i\,\omega_0$ ($\omega_0 > 0$) and no other eigenvalues of the form $\pm ik\omega_0$ ($k \in \mathbb{Z}$, $k \neq \pm 1$). Then the vectorfield f has a smooth R-invariant 2-dimensional locally invariant manifold containing x_0 and foliated by a 1-parameter family of symmetric periodic orbits. As one moves along this family of periodic orbits towards the equilibrium the minimal period tends to $T_0 := 2\pi/\omega_0$.* □

The situations described by the theorems 1 and 2 are robust under perturbations: if in a parametrized family of Hamiltonian (respectively reversible) vectorfields the conditions of Theorem 1 (respectively Theorem 2) are satisfied for a certain value λ_0 of the parameter then they remain satisfied for all nearby values of the parameter. The reason for this is that if $\mu_0 \in \mathbb{C}$ is an eigenvalue of A_0 then so is $-\mu_0$; as a consequence the simple purely imaginary eigenvalues whose existence was assumed in the foregoing theorems cannot move off the imaginary axis when the system is perturbed. However, in parametrized families of Hamiltonian or reversible systems it is possible to find in a generic way equilibria for which the linearization has a pair of purely imaginary eigenvalues for which the conditions of Theorems 1 and 2 are not satisfied, because either these eigenvalues are not simple, or because the nonresonance condition is not satisfied, or both. A well known example is that of a so-called *Krein instability* (also called a 1 : 1-*resonance* or a *Hamiltonian Hopf bifurcation*) in a one-parameter family of Hamiltonian systems: in their simplest form the hypotheses are that there is an equilibrium (say at $x = 0$) at which the linearization $A_\lambda := D_x X_H(0, \lambda)$ has for small $\lambda < 0$ two pairs of simple purely imaginary eigenvalues close to each other which merge for $\lambda = 0$ into a single pair of non-semisimple purely imaginary eigenvalues and split off the imaginary axis for $\lambda > 0$. An application of Theorem 1 shows that for fixed small $\lambda < 0$ the system has two one-parameter families of periodic orbits emanating from the equilibrium $x = 0$; the question arises what happens to these periodic orbits as λ passes through zero and becomes positive.

The answer to these question depends on some third order coefficient in the normal form of the vectorfield $X_H(\cdot, 0)$, i.e. on some fourth order coefficient in the normal form of the Hamiltonian $H(\cdot, 0)$. Generically (when considering one-parameter problems as described above) this coefficient is non-zero; depending on its sign we have either an *elliptic* or a *hyperbolic* bifurcation. In the elliptic case the two families of periodic orbits which emanate from the equilibrium for $\lambda < 0$ are connected and form one single branch which at both sides tends to the equilibrium; we call this a *local branch*. As λ increases towards zero this local branch shrinks and is absorbed by the equilibrium for $\lambda = 0$. For $\lambda \geq 0$ there are no nontrivial periodic orbits nearby the equilibrium. In the hyperbolic case we have the following scenario. For $\lambda < 0$ the two families of periodic orbits emanating from the equilibrium are not connected to each other (at least not locally); we say that we have two *global branches*. For $\lambda = 0$ these two global branches become at the equilibrium tangent to each other; for $\lambda > 0$ they detach from the equilibrium and merge into one single branch of periodic orbits which no longer contains the equilibrium. A complete analysis of this Hamiltonian Hopf bifurcation can be found in [20].

The same bifurcation scenario as described above also appears at generic 1 : 1 - resonances in one-parameter families of conservative or reversible systems (see respectively [6] and [7]). The result can be extended to equivariant conservative or equivariant reversible systems (see [14] and [8]). It is also possible to consider situations where $k > 2$ pairs of purely imaginary eigenvalues come together and split off the imaginary axis under a change of parameters; such situations appear generically in $k - 1$-parameter families of conservative or reversible systems. An analysis of the bifurcation of periodic orbits at such k-fold resonances can be found in [6] and [7].

A further question which arises in the context of such resonances is about the stability of the periodic orbits appearing in these bifurcation scenario's. It is important to notice that if $\mu \in \mathbb{C}$ is a characteristic multiplier of a periodic orbit in a Hamiltonian or reversible system, then so is $1/\mu$; consequently a periodic orbit is called *stable* if all its multipliers are on the unit circle, and *unstable* if there are some multipliers off the unit circle. Taking into account only the critical multipliers it can be shown for the Hamiltonian Hopf bifurcation described above (see [20]) that in the hyperbolic case all periodic orbits appearing in the bifurcation scenario are stable; in the elliptic case the local branch which exists for $\lambda < 0$ is divided into three parts: the periodic solutions along the middle part are unstable, those along the two outer parts (adjacent to the equilibrium) are stable. The same result also holds at a 1 : 1-resonance in reversible systems (see [4] and [9]); here the transition points between stable and unstable solutions along the local branch in the elliptic case are sometimes called *Eckhaus points*. At these Eckhaus points there can be secondary bifurcations, in particular of orbits homoclinic to periodic orbits (again, see [4]). Finally, the stability of periodic orbits near a 3-fold resonance in reversible systems will be discussed in some forthcoming paper [9].

There are several tools available for studying the bifurcation of periodic orbits at resonances in Hamiltonian or reversible systems; the most popular ones are the Liapunov-Schmidt reduction and normal form theory. We conclude this section by describing a general type of reduction result which can (and has) been used for analyzing the type of resonances considered here. More details and proofs can be found in [17] and [5]. These proofs are based on a combined use of normal form theory and the Liapunov-Schmidt reduction; however, the reduction result can be used directly, without going into the details of either of these methods (see [6] and [7] for some examples).

Consider a system

$$\dot{x} = f(x, \lambda), \tag{2.1}$$

where the vectorfield $f : \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}^{2n}$ is either Hamiltonian or reversible, and satisfies $f(0, \lambda) = 0$ for all λ. We are then interested in solving the following problem:

(**P**) Find, for all (λ, T) near a given $(\lambda_0, T_0) \in \mathbb{R}^m \times]0, \infty[$, all sufficiently small T-periodic solutions of $(2.1)_\lambda$.

Let $A_0 := D_x f(0, \lambda_0)$ be the linearization of $f(\cdot, \lambda_0)$ at the equilibrium in the origin, and assume that A_0 is nonsingular, such that there is no bifurcation of equilibria at $\lambda = \lambda_0$. Let $A_0 = S_0 + N_0$ be the Jordan decomposition of A_0 into its semisimple and nilpotent parts (i.e. S_0 is semisimple, N_0 is nilpotent, and $S_0 N_0 = N_0 S_0$). Next we introduce the so-called *reduced phase space* for our problem; this is a subspace of $\mathbb{R}^{2n}$ defined by

$$U := \ker \left(e^{S_0 T_0} - I \right). \tag{2.2}$$

There exists a natural S^1-action on U, generated by $S := S_0|_U$ and explicitly given by

$$\varphi \in S^1 \cong \mathbb{R}/T_0\mathbb{Z} \longmapsto e^{S\varphi} \in \mathcal{L}(U). \tag{2.3}$$

Also, the space U is even-dimensional and invariant under J or R depending on whether f is Hamiltonian or reversible; therefore it makes sense to talk about a Hamiltonian (respectively reversible) vectorfield on U.

We have then the following reduction result.

Theorem 3. *Under the foregoing conditions there exists for each (λ, T) near (λ_0, T_0) a one-to-one correspondence between the small T-periodic solutions of $(2.1)_\lambda$ and the small T-periodic solutions of a* reduced equation

$$\dot{u} = f_r(u, \lambda), \tag{2.4}$$

where the reduced vectorfield $f_r : U \times \mathbb{R}^m \to U$ *has the following properties:*

(1) $f_r(0, \lambda) = 0$ *for all* λ, *and* $D_u f_r(0, \lambda_0) = S + N$, *where* $N := N_0|_U$;
(2) f_r *is Hamiltonian or reversible, depending on whether f is Hamiltonian or reversible;*
(3) f_r *is S^1-equivariant, i.e. we have*

$$f_r(e^{S\varphi}u, \lambda) = e^{S\varphi} f_r(u, \lambda), \qquad \forall \varphi \in S^1; \tag{2.5}$$

Moreover, all small T-periodic solutions of $(2.4)_\lambda$ have the form

$$\tilde{u}(t) = e^{(1+\sigma)St}u \tag{2.6}$$

with $u \in U$ small and $T = T_0/(1+\sigma)$. □

An analogous result holds for conservative systems, under appropriate non-degeneracy conditions for the first integral. The last conclusion of Theorem 3 combined with the S^1-equivariance of f_r shows that in order to obtain the bifurcation picture for our problem (**P**) we have to study the *determining equation*

$$(1+\sigma)Su = f_r(u, \lambda) \tag{2.7}$$

for (u, λ, σ) near $(0, \lambda_0, 0)$. It is shown in [17] and [5] how the reduced vectorfield f_r can be calculated or approximated by bringing the original vectorfield f into normal form. It should be emphasized that although (2.7) is a finite-dimensional equation it is in general not yet the *bifurcation equation* for our problem (**P**) since its linearization at $(u, \lambda, \sigma) = (0, \lambda_0, 0)$ is not identically zero but gives the equation $Nu = 0$; however, when the nilpotent operator N is known it is fairly simple to deduce the bifurcation equations from (2.7).

3 Branches terminating at homoclinics

When moving along a branch of periodic orbits in a Hamiltonian, conservative or reversible system it is possible that the period tends to infinity while the orbit itself remains bounded; the limiting orbit may then for example be a homoclinic orbit. Examples of such *homoclinic period blow-up* are well known for one-degree-of-freedom Hamiltonian systems, i.e. when $n = 1$ in (1.1). Consider for example the phase portrait for the Hamiltonian system with Hamiltonian $H(y, z) = 1/2z^2 + y^3 - y^2$; this system has two equilibria, a center and a saddle; the periodic orbits which originate at the center terminate in a period blow-up at an orbit homoclinic to the saddle. In [15] it is shown that this type of behavior is typical near (symmetric) homoclinic orbits in conservative or reversible systems, whatever their dimension. In this section we briefly describe the main result of [15].

We consider a system

$$\dot{x} = f(x). \tag{3.1}$$

In the conservative case we assume that $x \in \mathbb{R}^n$, that $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, and that there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $DH(x) \cdot f(x) = 0$ for all $x \in \mathbb{R}^n$; moreover it is assumed that:

(**C**) (i) there exists an orbit γ_0 of (3.1) which is homoclinic to a hyperbolic equilibrium $x_0 \in \mathbb{R}^n$;

(ii) the homoclinic orbit γ_0 is non-degenerate, i.e.

$$\dim\,(T_y W^s(x_0) \cap T_y W^u(x_0)) = 1, \qquad \forall y \in \gamma_0, \tag{3.2}$$

where $W^s(x_0)$ and $W^u(x_0)$ denote the stable (respectively unstable) manifold of x_0;

(iii) $DH(y_0) \neq 0$ for some $y_0 \in \gamma_0$.

These hypotheses are robust under perturbations and imply a period blow-up at γ_0; more precisely:

Theorem 4. *Under the assumptions* (**C**) *we have that* $\gamma_0 \cup \{x_0\}$ *forms the limit of a one-parameter family of periodic orbits along which the minimal period* T *tends to infinity as one approaches the homoclinic orbit.* □

When the system (3.1) is reversible (see Section 1) one has to assume that the homoclinic orbit is symmetric, and then necessarily also the limiting equilibrium is symmetric. Such symmetric homoclinic orbits have a unique intersection point with $\mathrm{Fix}\,(R)$. Also, if $x_0 \in \mathrm{Fix}\,(R)$ is a symmetric and hyperbolic equilibrium, then both the stable manifold $W^s(x_0)$ and the unstable manifold $W^u(x_0)$ have dimension n, since $W^u(x_0) = R(W^s(x_0))$. This allows us to formulate our hypotheses for the reversible case as follows:

(**R**) (i) the system (3.1) is reversible and has a symmetric orbit γ_0 (i.e. $R(\gamma_0) = \gamma_0$) which is homoclinic to a symmetric and hyperbolic equilibrium $x_0 \in \mathrm{Fix}\,(R)$;

(ii) γ_0 is an *elementary* homoclinic orbit, which means that $W^s(x_0)$ and $\mathrm{Fix}\,(R)$ intersect transversely at the unique intersection point of γ_0 and $\mathrm{Fix}\,(R)$.

Again these hypotheses are robust under perturbations, and they imply a homoclinic period blow-up along a family of *symmetric* periodic orbits.

Theorem 5. *Under the assumptions* (**R**) *we have that* $\gamma_0 \cup \{x_0\}$ *forms the limit of a one-parameter family of symmetric periodic orbits along which the minimal period* T *tends to infinity as one approaches the homoclinic orbit.* □

The proofs of these theorems as given in [15] is based on a simplified form of Lin's method (see [10]); this method has recently become quite popular for the study of bifurcations near homoclinics (see e.g. the recent work of B. Sandstede).

4 Subharmonic branching

In the foregoing sections we have seen how branches of periodic orbits in Hamiltonian or reversible systems can originate at equilibria or terminate at homoclinics. In this section we discuss some elementary "branching phenomena" which can occur along branches of periodic orbits; we also describe a reduction result for mappings (analogous to Theorem 3) which can be used to study such branchings. For the sake of simplicity we will restrict here to Hamiltonian systems, although most of the results have analogues for reversible systems (see e.g. [16] for a study of subharmonic branching in reversible systems).

To start consider a Hamiltonian system

$$\dot{x} = X_H(x) = J\nabla_x H(x), \tag{4.1}$$

with $x \in \mathbb{R}^{2n}$ and $H : \mathbb{R}^{2n} \to \mathbb{R}$ smooth. Let γ_0 be a given (nontrivial) periodic orbit of (4.1), $x_0 \in \gamma_0$ and $h_0 := H(x_0)$. As described in the Introduction we can then construct a one-parameter family of (restricted) Poincaré maps

$$P_h : \Sigma_h \longrightarrow \Sigma_h, \tag{4.2}$$

well defined for $h \in \mathbb{R}$ close to h_0 (see Section 1 for the notations). Fixed points of P_h correspond to periodic orbits of (4.1) close to γ_0, periodic orbits of P_h correspond to so-called *subharmonic solutions* of (4.1), i.e. periodic solutions whose orbit remains in a neighborhood of γ_0 but whose minimal period is close to an integer multiple of of γ_0. So the study of periodic orbits near γ_0 leads to an analysis of the bifurcation of fixed points and periodic points from the fixed point x_0 of P_{h_0}. The following properties of Σ_h and P_h are crucial for this analysis.

Lemma 6. *We have for each* h *near* h_0 *that* Σ_h *is a* $2(n-1)$*-dimensional symplectic submanifold of* $\mathbb{R}^{2n}$*, and* P_h *is a symplectic diffeomorphism.* □

For a proof see e.g. [13] or [19]. Using the classical Darboux theorem (see [1] or [19]) this Lemma implies that the family of mappings P_h ($h \in \mathbb{R}$) can be identified with a one-parameter family of symplectic diffeomorphisms on a fixed *symplectic vectorspace* (V, J) with $\dim V = 2(n-1)$; this means that $J \in \mathcal{L}(V)$ is anti-symmetric with respect to some scalar product on V and satisfies $J^2 = -I_V$, while the diffeomorphisms $P_h : V \to V$ are such that

$$DP_h(x)^T J DP_h(x) = J, \qquad \forall x \in V. \tag{4.3}$$

In this identification the base point x_0 at which we constructed the Poincaré map corresponds to the origin of V; hence we have $P_{h_0}(0) = 0$. The eigenvalues of $DP_{h_0}(0)$ are the nontrivial characteristic multipliers of the periodic orbit; observe that because of the symplectic structure 1 will always be a multiplier with at least multiplicity 2. Generically 1 will be a multiplier with multiplicity equal to 2, and in that case 1 will not be an eigenvalue of $DP_{h_0}(0)$ and the fixed point of P_{h_0} will persist for nearby values of h. Therefore we can (possibly after an appropriate translation) assume that

$$P_h(0) = 0, \qquad \forall h \in \mathbb{R}. \tag{4.4}$$

The fixed point set $\{(0, h) \mid h \in \mathbb{R}^m\}$ corresponds to the branch of periodic solutions of (4.1) which we discussed in Section 1.

Now let us consider the eigenvalues of $DP_h(0) \in \mathcal{L}(V)$. Setting $x = 0$ in (4.3) it is easy to show that if $\mu \in \mathbb{C}$ is an eigenvalue of $DP_h(0)$ then so are $1/\mu$, $\bar{\mu}$ and $1/\bar{\mu}$. It follows that if $DP_{h_0}(0)$ has a pair of *simple* eigenvalues $\{\mu, \bar{\mu}\}$ on the unit circle (i.e. $|\mu| = 1$ and $\mu \neq \pm 1$), then the continuation of these eigenvalues stays on the unit circle for all h near h_0. Hence we expect to see many values of h for which $DP_h(0)$ has a pair of simple eigenvalues which are *roots of unity*, i.e. of the form $\exp(\pm 2\pi i p/q)$, with $0 < p < q$ and $\gcd(p, q) = 1$. This means that the linearization $DP_h(0)$ has q-periodic points, and hence there is a possibility that in the family of diffeomorphisms P_h we see a bifurcation of q-periodic points from the fixed point at 0. This in turn would mean that we have bifurcation of subharmonic solutions near γ_0 for our original Hamiltonian system (4.1).

We now describe a general reduction result which is very useful in studying the bifurcation of periodic points from fixed points in families of symplectic mappings and which forms an analogy for mappings of what we found in Theorem 3 for vectorfields. The proof can be found in [2], and a similar result for reversible mappings will be given in [3].

Let (V, J) be a symplectic vectorspace and $\Phi : V \times \mathbb{R}^m \to V$ a parametrized family of symplectic diffeomorphisms, i.e. we have

$$D\Phi_\lambda(x)^T J D\Phi_\lambda(x) = J, \qquad \forall x \in V, \forall \lambda \in \mathbb{R}^m, \tag{4.5}$$

with $\Phi_\lambda := \Phi(\cdot, \lambda)$ for $\lambda \in \mathbb{R}^m$. We also assume that $\Phi_\lambda(0) = 0$ for all λ, and (taking $\lambda = 0$ as a critical parameter value) we set $A_0 := D\Phi_0(0)$. Given an integer $q \geq 1$ we then consider the following problem:

($\mathbf{P}_q$) Find, for all small λ, all small q-periodic points of Φ_λ.

To solve ($\mathbf{P}_q$) we have to solve the equation

$$\Phi_\lambda^q(x) = x \tag{4.6}$$

for all (x, λ) near $(0, 0)$. We notice that this equation has an *implicit* $\mathbb{Z}_q$*-symmetry*: if, for a given λ, $x \in V$ is a solution of (4.6), then so are $\Phi_\lambda(x)$, $\Phi_\lambda^2(x)$, $\ldots$, $\Phi_\lambda^{q-1}(x)$ and $\Phi_\lambda^q(x) = x$. The result which follows will make this implicit symmetry explicit, so that in applications it can be used to simplify the equations. Let $A_0 = S_0 + N_0$ be the Jordan decomposition of A_0 into its semisimple part S_0 and its nilpotent part N_0, and define the *reduced phase space* U by

$$U := \ker (S_0^q - I) \tag{4.7}$$

One can then show that S_0 is a symplectic linear operator on V, that U is a symplectic subspace of V, and that $S := S_0|_U \in \mathcal{L}(U)$ generates a natural symplectic $\mathbb{Z}_q$-action on U.

Theorem 7. *For each sufficiently small λ there exists a one-to-one correspondence between the small q-periodic points of Φ_λ and the small q-periodic points of a* reduced mapping $\Phi_{r,\lambda}$, *where*

$$\Phi_r : U \times \mathbb{R}^m \longrightarrow U$$

has the following properties:

(i) *$\Phi_r(0, \lambda) = 0$ for all λ, and $D_u\Phi_r(0, 0) = S + N$, where $N := N_0|_U$;*

(ii) *$\Phi_{r,\lambda}$ is a symplectic diffeomorphism on U, for each λ;*

(iii) *Φ_r is $\mathbb{Z}_q$-equivariant, i.e. we have*

$$\Phi_r(Su, \lambda) = S\Phi_r(u, \lambda). \tag{4.8}$$

Moreover, all small q-periodic orbits of $\Phi_{r,\lambda}$ are also $\mathbb{Z}_q$-orbits, i.e. they can be found by solving the $\mathbb{Z}_q$-equivariant determining equation

$$\Phi_r(u, \lambda) = Su. \tag{4.9}$$

Finally, the reduced mapping Φ_r can be approximated up to any finite order by bringing the original mapping Φ into normal form. □

We conclude this section with a brief indication on how the reduction result of Theorem 7 can be used to prove a classical result of Meyer [11] on the bifurcation of periodic points in symplectic mappings. Again the details of our approach can be found in [2]. We take $m = 1$ in the foregoing and fix some $q \geq 3$. We also assume the following:

(a) A_0 has a pair of simple eigenvalues $\exp(\pm 2\pi ip/q)$, with $0 < p < q$ and $\gcd(p, q) = 1$;

(b) A_0 has no other eigenvalues μ such that $\mu^q = 1$.

Then dim $U = 2$ and we can identify U with the complex plane, such that the reduced mapping Φ_r given by Theorem 7 now becomes a mapping from $\mathbb{C} \times \mathbb{R}$ into $\mathbb{C}$. The $\mathbb{Z}_q$-equivariance (4.7) then takes the form

$$\Phi_r(e^{2\pi ip/q}z, \lambda) = e^{2\pi ip/q}\Phi_r(z, \lambda), \qquad \forall (z, \lambda) \in \mathbb{C} \times \mathbb{R}, \tag{4.10}$$

while the determining equation (4.9) becomes

$$\Phi_r(z, \lambda) = e^{2\pi ip/q}z. \tag{4.11}$$

It was shown in [18] that (4.10) implies that Φ_r must have the form

$$\Phi_r(z, \lambda) = \phi_1(z, \lambda)\, z + \phi_2(z, \lambda)\, \bar{z}^{q-1}, \tag{4.12}$$

with the functions $\phi_i : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ $(i = 1, 2)$ such that

$$\phi_i(e^{2\pi ip/q}z, \lambda) = \phi_i(z, \lambda) = \phi_i(\bar{z}, \lambda), \qquad \forall (z, \lambda) \in \mathbb{C} \times \mathbb{R},\ i = 1, 2. \tag{4.13}$$

Since Φ_r is symplectic it is also area-preserving, which in combination with (4.12) and (4.13) implies that

$$|\phi_1(z, \lambda)| = 1 + \mathrm{O}(|z^q|).$$

Using polar coordinates and assuming some generically satisfied conditions one can then solve the determining equation (4.11). The result is that (4.11) has $2q$ branches of nontrivial solutions, each parametrized by the amplitude ρ of z, and of the form

$$\{(\rho e^{i(\theta_i^*(\rho)+2\pi jp/q)}, \lambda_i^*(\rho)) \mid 0 < \rho < \rho_0\}, \qquad (0 \le j \le q-1,\ i = 1, 2)$$

with $\theta_2^*(0) = \theta_1^*(0) + \pi/q$, $\lambda_i^*(\rho) = \mathrm{O}(\rho^2)$ $(i = 1, 2)$ and $\lambda_2^*(\rho) = \lambda_1^*(\rho) + \mathrm{O}(\rho^{q-2})$. So there are two branches of periodic orbits which bifurcate at $\lambda = 0$ from the fixed point at the origin in the family of symplectic diffeomorphisms Φ_λ $(\lambda \in \mathbb{R})$.

When we apply the foregoing result to the family P_h $(h \in \mathbb{R})$ of Poincaré maps discussed in the beginning of this section we conclude the following: when at some point along a one-parameter branch of periodic orbits of (4.1) the nontrivial characteristic multipliers satisfy the conditions (a) and (b) (for some $q \ge 3$) then at that point two branches of subharmonic solutions will bifurcate from the first branch. The higher the value of q, the closer to each other these two branches will be. The bifurcating subharmonic solutions will (next to the double multiplier 1) have two multipliers close to 1; along one of the two branches these "critical multipliers" are on the real axis, along the other branch they are on the unit circle.

5 Subharmonic cascades

In this last section we briefly indicate an interesting but still largely open problem. Consider again, as in the foregoing section, a one-parameter branch of periodic orbits of the Hamiltonian system (4.1). Assume that along part of this branch there are some simple multipliers on the unit circle. Then there will generically be an infinite number of points along the branch where some multipliers are roots of unity and where we will have bifurcation of two branches of subharmonic solutions. At most of these bifurcation points the value of q will be high, and hence the bifurcating subharmonics will have large periods. Now concentrate on one such branching point; as indicated at the end of Section 4 the multipliers along one of the two bifurcating branches will be on the unit circle and close to 1. Hence, applying again the same results, we will find along this secondary branch an infinite number of points where two branches of subharmonics bifurcate; since the critical multipliers along the secondary branch are close to 1 the subharmonics bifurcating from this branch will have very high periods (corresponding to very large values of q). Iterating this argument we obtain a *cascade* of subharmonic branchings, all in the same fixed Hamiltonian system (4.1). A similar argument can be given for reversible systems. It leads to a very rich and complicated structure for the set of periodic orbits of Hamiltonian or reversible systems, and it would certainly be interesting to understand this structure in a more global way.

The methods described in the foregoing sections do not allow such global study since they are local (near each of the branching points) and they concentrate on solutions with a given (approximate) period. One will need a different approach to answer such questions as: (i) is there any self-similarity in such cascades? (ii) can one use renormalization techniques? (iii) are there any universal constants? In some very particular cases (mainly concentrating on period-doubling) some of these questions have been answered by a number of authors such as M. Feigenbaum and R. MacKay. In the reversible case there is some recent contribution by J. Roberts and J. Lamb ([12]). But to a large extend the problem remains open.

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