EQUADIFF 2

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Axiomatization of differential equation theory

In: Valter Šeda (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Bratislava in September 1966. Slovenské pedagogické nakladateľstvo, Bratislava, 1967. Acta Facultatis Rerum Naturalium Universitatis Comenianae. Mathematica, XVII. pp. 163--172.

Persistent URL: http://dml.cz/dmlcz/700220

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ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

1. Ordinary Differential Equations

AXIOMATIZATION OF DIFFERENTIAL EQUATION THEORY

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This lecture is an attempt to motivate, describe and justify an axiomatic treatment of several basic portions of differential equation theory, or more precisely, of the initial value problem for ordinary differential equations.

1. It seems that a situation in mathematics is judged ripe for axiomatization (non-categorial, i.e. possessing non-isomorphic realizations) if, in loose terms, there is a number of independent subjects which exhibit common or similar properties; and second, if it is also recognized, explicitly or not, that significant portions of the development of these subjects stem from these common properties rather than from the individual specific nature of the subjects themselves.

I claim that such a situation has evolved in connection with differential equations. The basic subject there is the theory of ordinary differential equations in the classical sense,

$$\frac{\mathrm{d}x}{\mathrm{d}\vartheta} = f(x, \vartheta) \qquad \text{with } x \in R^n,\ \vartheta \in R^1, \tag{1}$$

and with $f : R^{n+1} \to R^n$ continuous. However, one frequently meets with similar equations in which the right-hand term f exhibits various types of discontinuity (e.g. a discontinuous forcing term or feed-back or coefficients); and also with the less closely related concepts of difference- and functional-differential equations, differential inequalities and equations in contingents. Next, significant generalizations are obtained by relaxing the requirement on the euclidean structure of the phase space in which the equations are to act; e.g., on replacing euclidean n-space R^n in (1) by a differential manifold (cf. differential equations on the torus, etc.), or even by various abstract spaces familiar from functional analysis (cf. ordinary differential equations in function spaces, to treat some partial differential equations). As slightly less important members of this family, one may mention the implicit differential equations, some integro-differential equations, and the finite difference equations.

Separately, each of these theories is, of course, perfectly adequate to its own professed main problem; however, they are intimately but informally related, using a similar terminology and arsenal of primitive notions. Thus in each case, a fundamental concept is that of an appropriately defined solution to an initial value problem; and in each case it is felt necessary to carry out, to some extent at least, a programme of development on the lines of classical ordinary differential equation theory in R^n. As a trivial example, in the case of difference-differential equations one is not surprised at finding an existence theorem proved via the Banach contraction mapping theorem; indeed, rather the opposite situation would be surprising.

To proceed one step further, I believe that hypothetical further theories would be held to belong to differential equation theory only if they conform in a similar sense as do those listed above, i.e. if they exhibit a reasonable recurrence of the fundamental properties and results. To express this even more strongly, I wish to suggest that most differential equationists actually possess an informal — and possibly unrecognized — concept of a general theory of differential equations, of which the theories mentioned previously are special cases.

The advantages to be gained from an axiomatic approach are then exactly those which apply to the axiomatic treatment of any informal theory: generality, perspicuity and economy of results and methods, and, as a secondary effect, in a number of cases even significant simplification or extension.

All this is, in my opinion, sufficient motive to attempt the explicit formulation of a general theory.

2. The first task then is to select a suitable general concept, capable of representing all the objects studied in differential equation theory; the term chosen was that of a process, [3]. As is often the case, this concept was not arrived at in a single stage, but represents the final step of what now appear to be partial axiomatizations of the notion of a differential equation. These include the dynamical systems (A. A. MARKOV, **1931**; [4, chap. *V*]), the flows (origin unknown), the "general systems" of ZUBOV (**1957**; [6, chap. *IV*]), and the local dynamical systems (HÁJEK, **1964**; [1]). These correspond to, or rather generalize, differential equations under various combinations of requirements on autonomness, unicity and prolongability of solutions; and in this sense, the processes correspond to differential equations, without any extraneous assumptions.

To introduce the concept of a process, first consider the basic model, viz. a classical ordinary differential equation (1). Explicitly, the assumptions are that f is a continuous partial map $R^{n+1} \to R^n$ with $D = \mathbf{domain}\, f$ open in R^{n+1}; and the solutions of (1) are defined as those partial maps $s : R^1 \to R^n$ with **domain** s an interval, which satisfy (1) in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} s(\vartheta) = f(s(\vartheta), \vartheta) \qquad \text{for all} \qquad \vartheta \in \mathbf{domain}\, s.$$

Of course, all this is easily carried over to differential equation on differentiable n-manifolds. With this differential equation we shall associate a process p. This is the relation in R^{n+1} determined as follows: $(x, \alpha) \in R^n \times R^1$ is to be in the relation p to an $(y, \beta) \in R^n \times R^1$, and this is written as $(x, \alpha)\, p(y, \beta)$, if and only if $\alpha \geq \beta$ and there exists a solution s of (1) with

$$x = s(\alpha), \qquad y = s(\beta)$$

(this includes the requirement that the interval domain of s contains both α, β).

It can be shown rather easily that the relation p describes the originally given equation (1) completely. This established a general method of assigning a process — to be called a differential process — to a differential equation. Similarly, there is a canonic method of assigning processes to discontinuous differential equations, to functional-differential equations, etc. (two further cases are discussed below). The processes obtained in this manner are all special cases of a single general concept which will now be described explicitly.

It will be said that p is a *process* in P over R iff P is a set (the phase space), R is a subset of R^1 (the set of admissible time instants), and p is a relation in $P \times R$ with the following three properties:

0^0 If $(x, \alpha)\, p\, (y, \beta)$ then $\alpha \geq \beta$.

1^0 If $(x, \alpha)\, p\, (y, \beta)$ and $\alpha = \beta$ then also $x = y$ (the inital value property).

2^0 p is a transitive relation, i.e.

$$(x, \alpha)\, p\, (y, \beta) \qquad \text{and} \qquad (y, \beta)\, p\, (z, \gamma) \tag{2}$$

imply $(x, \alpha)\, p\, (z, \gamma)$; also, in partial converse, whenever $(x, \alpha)\, p(z, \gamma)$ and $\alpha \geq \beta \geq \gamma$ in R, there exists an $y \in P$ with (2) (the compositivity property).

Occasionally a minor modification of this notation is more useful. Given objects p, P, R with property 0^0 as above, for each $\alpha \geq \beta$ in R define a relation ${}_\alpha p_\beta$ on P by letting

$$x\, {}_\alpha p_\beta\, y \qquad \text{iff} \qquad (x, \alpha)\, p(y, \beta). \tag{3}$$

Evidently p is completely determined by the indexed system of relations $\{{}_\alpha p_\beta \mid \alpha \geq \beta \text{ in } R\}$. Then 1^0 and 2^0 may be formulated more concisely:

1^0 ${}_\alpha p_\beta \subset I$ (the identity relation on P) for all $\alpha \in R$.

2^0 ${}_\alpha p_\beta \circ {}_\beta p_\gamma = {}_\alpha p_\gamma$ for all $a \geq \beta \geq \gamma$ in R.

Both these descriptions, using p and the ${}_{\alpha}p_{\beta}$, will be used, always invoking definition (3) automatically.

Returning to the (differential) processes associated with differential equations as described above, it is easily seen that 0^0 and 1^0 are satisfied automatically; and 2^0 follows from obvious properties of solutions of (1), namely from the fact that any interval-partialization of a solution is again a solution, and that the concatenation of (concatenable) solutions is a solution. Thus p is a process in R^n over R^1.

As a less immediate interpretation, consider a difference-differential equation with constant time lag

$$\frac{\mathrm{d}x}{\mathrm{d}\vartheta} = f(x(\vartheta - \tau), x(\vartheta), \vartheta), \tag{4}$$

given continuous $f : R^3 \to R^1$ and $\tau > 0$. For definiteness, the solutions of (4) are continuous maps $s : [\beta - \tau, \alpha] \to R^1$ for given $-\infty \leq \beta \leq \alpha \leq +\infty$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} s(\vartheta) = f(s(\vartheta - \tau), s(\vartheta), \vartheta) \qquad \text{for} \qquad \beta < \vartheta \leq \alpha$$

(with obvious modifications for the case of non-closed domains). It will be convenient to write x_λ for the λ-translate of a partial map $x : R^1 \to R^1$, so that $x_\lambda(\vartheta) = x(\vartheta + \lambda)$ whenever defined. The inital value problem for (4) is to find, to given $\beta \in R^1$ and continuous $y : [-\tau, 0] \to R^1$, a solution s of (4) as above, and satisfying $y \subset s_\beta$, i. e. such that $s(\vartheta) = y(\vartheta - \beta)$ for $\beta - \tau \leq \vartheta \leq \beta$. This situation may be usefully described by a process p in the function space $C^1[-\tau, 0]$ over R^1: For x, y in $C^1[-\tau, 0]$ and $\alpha \geq \beta$ in R^1 let $(x, \alpha)\, p(y, \beta)$ iff $x \subset s_\alpha$, $y \subset s_\beta$ for some solution s of (4). Again it is easily verified that this relation p satisfies axioms 0^0 to 2^0 and hence defines a process $C^1[-\tau, 0]$ over R^1; and that this process characterizes the original equation completely. Very similar constructions may be carried out more generally for functional-differential equations; not necessarily of retarded type, in n-space.

The final example concerns a one-dimensional partial differential equation

$$\frac{\partial u}{\partial \vartheta} = f\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial^2 u}{\partial \xi^2}, \xi, \vartheta\right) \tag{5}$$

with continuous $f : R^5 \to R^1$; consider the corresponding homogeneous boundary value problem in the strip $\{(\xi, \vartheta) \in R^2 : |\xi| \leq 1, \vartheta \geq 0\}$. The associated process p will act in the set P of all continuous functions on $[-1, 1]$ with zero end values. For x, $y \in P$ and $\alpha \geq \beta$ in R^1 one defines that $(x, \alpha)\, p(y, \beta)$ iff

$\beta \geq 0$ and there exists a solution u of (5) with zero boundary values and such that

$$u(\xi, \alpha) = x(\xi), \qquad u(\xi, \beta) = y(\xi) \qquad \text{for} \qquad |\xi| \leq 1.$$

Again, a similar construction may be carried out for higher orders, for more complicated domains and boundary conditions, and for systems of such equations.

3. It is now appropriate to show how several fundamental concepts may be carried over from differential equations to processes. Thus, assume given a process p in P over R. (In the envisaged applications, the set R of admissible time instants is either the real axis R^1, or the set C^1 of integers for processes with discrete time; the present formulation was designed to cover both situations.) A *solution* of p is defined as any partial map $s : R \to P$ with domain s an interval in R and such that $(s(\alpha), \alpha)\, p(s(\beta), \beta)$ for all $\alpha \geq \beta$ in domain s. For differential processes these are precisely the solutions of the equation in the usual sense.

The set of all pairs $(x, \alpha) \in P \times R$ such that $(x, \alpha)\, p(x, \alpha)$ will be denoted by D and termed the *domain* of p. Directly from the axioms, $(x, \alpha)\, p(y, \beta)$ implies that both (x, α), (y, β) are in D; thus essentially p concerns only the elements of $D \subset P \times R$. For the differential process associated with (1) this set D coincides with **domain** f.

The process p is said to have *unicity* iff $u_\vartheta p_\alpha x$ and $u'_\vartheta p_\alpha x$ always imply $u' = u$. The process p is termed global or said to have global existence (or indefinite prolongability) iff to any $(x, \alpha) \in D$ and $\vartheta \geq \alpha$ in R there exists an $u \in P$ with $u_\vartheta p_\alpha x$. Slightly more generally, to any $(x, \alpha) \in D$ one may assign a numerical characteristic $\varepsilon(x, \alpha)$, the *extent of existence* of p at (x, α), defined as

$$\varepsilon(x, \alpha) = \sup \{\vartheta \in R : u_\vartheta p_\alpha x \qquad \text{for some} \qquad u \in P\}.$$

Easily, $\alpha \leq \varepsilon(x, \alpha) \leq +\infty$. If $\alpha < \varepsilon(x, \alpha)$ one says that *local existence* obtains at (x, α) and in the opposite case (x, α) is called an *end-pair*. If $\varepsilon(x, \alpha) = +\infty$ one says that global existence obtains at (x, α), and in the opposite case (x, α) is said to have *finite escape time*.

The process p will be termed *stationary* (or autonomous) iff R is an additive subgroup of R^1 and, for all $\alpha \geq \beta$ and ϑ in R, ${}_\alpha p_\beta = {}_{\alpha+\vartheta} p_{\beta+\vartheta}$. In this case a point $x \in P$ is called *critical* iff $x_\vartheta p_\alpha x$ for all $\vartheta \geq \alpha$ in R. In the obvious manner one may define *cycles* with given primitive period, invariant sets, etc.

A real-valued function λ on $P \times R$ is called a Liapunov function for p if $(x, \alpha)\, p(y, \beta)$ implies $0 \leq \lambda(x, \alpha) \leq \lambda(y, \beta)$. (This definition can be generalized extensively.)

For differential processes, all these concepts assume their classical meaning;

thus they are the corresponding generalizations. Having determined the appropriate formulations of these concepts in the general situation, one may apply them automatically in the various special cases. Thus one has, e. g. the concept of critical points for stationary difference-differential and functional-differential equations. As a matter of fact, in the former case these had already been introduced, and agree with the present; to my knowledge, in the latter case these have not been studied.

There is one exception to this rule, concerning the concept of solutions. Thus a solution of the partial differential equation (5) in the customary sense is a real-valued function u of two real variables ξ, ϑ; and a solution of the associated process is a function-valued map s with the variable ϑ. However, one has an obvious one-to-one correspondence determined by

$$u(\xi, \vartheta) = (s(\vartheta))\,(\xi).$$

In the case of the difference-differential equation (4) the divergence is even more marked, but once again there is a one-to-one correspondence between the corresponding solutions.

This illustrates the assertion that the fundamental concepts from differential equation theory find adequate and natural generalizations within process theory. As concerns the methods, I have space only for an elementary example. It is well known that every differential equation (1) in R^n may be "made stationary" by passing to a different equation in R^{n+1}, namely the system

$$\text{(6)} \qquad \frac{\mathrm{d}x}{\mathrm{d}\vartheta} = f(x, \lambda), \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}\vartheta} = 1.$$

The relation between these is that the first n coordinates of any solution of (6) constitute a solution of (1), and conversely. This stationarization procedure appears in process theory also. Thus, let p be a process in P over $R = R^1$, say. Define a new process q in $Q = P \times R$ over R by setting, for (x, ξ), $(y, \eta) \in \in Q$ and $\alpha \geq \beta$ in R,

$$(x, \xi)\,{}_{\alpha}q_{\beta}(y, \eta) \quad \text{iff} \quad x\,{}_{\xi}p_{\eta}\,y \quad \text{and} \quad \xi - \alpha = \eta - \beta.$$

It is then easily verified that q is indeed a stationary process in Q over R, and that it has to p a relation corresponding precisely to that obtaining between (6) and (1).

In this example, to carry over the method from differential equations to processes, it was not necessary to assume anything concerning the nature of the phase space P; indeed, it could be any abstract set. However, in other cases one must introduce further requirements. Thus, e.g. in attempting to introduce the concept of limit points or of orbital stability for processes, it is necessary to employ notions describing the nearness of a set to a point;

slightly more precisely, to assume that some structure such as a topology for P has been given in advance. Then it may (but need not) be necessary to require that the process p itself be compatible in some sense with the given structure on the phase space (that p be a "continuous" process). As reasonable candidates for interesting structures, the following seem to present themselves:

structure for P	compatible p
topology (or uniformity, metric, differential, etc.)	continuous
group	additive
linear space	linear
differential	differential

(combinations of these are also interesting; e.g. BANACH spaces and continuous linear processes).

As an example of this group of definitions, a process p in a linear space L over R is termed *linear* iff

$$x_\alpha p_\beta y, \qquad x'_\alpha p_\beta y', \qquad \lambda \in R^1 \qquad \text{imply} \qquad (x + \lambda x')_\alpha p_\beta (y + \lambda y').$$

Two linear processes p and p' in a HILBERT space H over R are called *adjoint* iff

$$x_\alpha p_\beta y, \qquad x'_\alpha p_\beta y' \qquad \text{imply} \qquad (x, x') = (y, y')$$

with (x, x') denoting the scalar product.

The definition of continuity of a process (in a topological space) is considerably more involved. However, in the not too special case of processes with global existence and unicity, this is quite straightforward. Assume given such a process p in a topological space T over R (the latter is to inherit the natural topology from $R^1 \supset R$). Unicity then yields that in any relation $(x, \alpha)\, p (y, \beta)$, the point $x \in T$ is uniquely determined by (α, y, β), thereby defining a partial map

$$t : R \times T \times R \to T, \qquad x = t(\alpha, y, \beta) \qquad \text{iff} \qquad (x, \alpha)\, p (y, \beta).$$

(This map t is called the global *flow* associated with p.) Then the process p is called *continuous*, or compatible with the given topology for T, iff the corresponding partial map t is continuous in the customary sense. In greater detail, the requirement is that

$(x_i, \alpha_i)\ p\ (y_i, \beta_i)$, $(\alpha_i, y_i, \beta_i) \longrightarrow (\alpha, y, \beta)$ in $R \times T \times R$, $(x, \alpha)\ p\ (y, \beta)$ imply $x_i \to x$ in T.

To define stability or recursive motions, a topology on the phase space is insufficient, since one must treat the nearness of two sets rather than that of a set to a point; and it is necessary to assume that the phase space is endowed with some structure such as a proximity or uniformity or metric. As concerns the process studied, in the differential case it is not necessary to assume that it be uniformly continuous (or distance preserving, etc.), but only continuous. Therefore one does not require compatibility between the process and e.g. the metric structure, but it still may be useful to impose compatibility with the topology induced by the metric. Thus one studies continuous processes on uniform spaces, on differential manifolds, etc.

This concept of continuity of processes is surprisingly versatile, allowing many classical results to be carried over to the more general situation. Thus e.g. MASSERA's first theorem on periodic solutions in R^1 [5, p. 445] can be transferred bodily, including its proof. To illustrate a more complicated case with a definitely non-trivial transfer to processes, the POINCARÉ—BENDIXSON theory of limit points and cycles for autonomous differential equations in the plane can be extended to stationary processes with unicity and local existence (the dynamical systems) on a large class of 2-manifolds [2].

Perhaps it is not surprising that the axioms 0^0 to 2^0 still permit some rather pathological objects as processes. Thus, consider the following very reasonable property: a process p is called *solution-complete* if all p-related pairs can be joined by a solution, i.e. iff $(x, \alpha)\, p(y, \beta)$ implies that there is a solution s of p with $x = s(\alpha)$, $y = s(\beta)$. Evidently the differential processes, etc., are all solution-complete. However, there do exist otherwise reasonable processes which are not solution-complete. Indeed, let P be the set of all real rationals, and for (x, α), $(y, \beta) \in P \times R^1$ with $\alpha \geq \beta$ let

$$(x, \alpha)\, p(y, \beta) \qquad \text{iff} \qquad \begin{array}{ll} 0 < x - y < \alpha - \beta & \text{in case } \alpha > \beta, \\ x = y & \text{in case } \alpha = \beta. \end{array}$$

Then p is a process in P over R^1 (in verifying the second part of requirement 2^0 use the fact that P is dense in R^1) indeet, p is closely related to the differential inequality $0 < \mathrm{d}x/\mathrm{d}\vartheta < 1$. Second, all solutions of p are continuous, since they have LIPSCHITZ constant 1; thus they are rational-valued continuous functions with interval domains, and hence all solutions are constant. But evidently the process has no (non degenerate) constant solutions at all. Therefore no distinct p-related pairs (x, α), (y, β) can be joined by any solution, i.e. p is not solution-complete.

4. The preceding section suggests that much of differential equation theory can be adequately represented within the wider setting of process theory. However, to justify the introduction and further study of processes, there are two further questions which should be answered satisfactorily. First, is

process theory capable of an autonomous development, of obtaining interesting results within itself, or is it merely an arid generalization or medium of reformulation, in which all the impetus is due to the classical underlying theories. And second, does process theory yield new results outside itself, i.e. can one obtain, via the processes, hereto unknown results formulatable in terms of only, e.g., differential equations.

Naturally, a decisive answer will not be available until much later; but even at this early stage of development, I have the impression that the answer to both these questions is affirmative. As examples to the first, consider the following two results (the formulations are somewhat loose):

Theorem. *Every solution-complete process can be represented, in a certain minimal and canonic fashion, as having been obtained from a process with unicity by identifying some elements in its domain; indeed, this representability characterizes solution-complete processes.*

Incidentally, the construction of the corresponding process with unicity seems closely related to that for difference-differential equations described earlier.

Differentiable representation theorem. *Every continuous process on a differentiable manifold P over R^1 with unicity and local existence and with domain open in $P \times R^1$ can be homeomorphically represented as corresponding to a differential equation.*

This shows, in particular, that at least for the indicated type of process, the axioms 0^0 to 2^0 exactly adequate, that no further independent axioms can be added. As concerns the second question, of the direct effect of process theory on differential equation theory, the results obtained are far less decisive and spectacular. However, one has the following

Proposition. *Let p and q be adjoint linear processes (in a* Hilbert *space). If p has global existence, then q has unicity in the negative direction; in particular, if p has global existence in both directions, then q has unicity in both directions.*

This has some interesting applications. Some time ago Dr. Karták studied linear homogeneous equations in n-space,

$$\frac{\mathrm{d}x}{\mathrm{d}\vartheta} = A(\vartheta)\, x \tag{7}$$

with continuity of the matrix A weakened to Newton-integrability (i.e. $A(\vartheta) = \frac{\mathrm{d}}{\mathrm{d}\vartheta} B(\vartheta)$ pointwise for some matrix B). Recently, he solved positively the general existence problem. Since change of orientation and passage to the adjoint equation in (7) yield equations which again have Newton-integrable coefficients (to which this existence theorem applies), the proposition above answers the general unicity problem positively. Obviously one even has a more general assertion: For every class of linear equations closed with respect to orientation change and formation of adjoints, global existence implies unicity.

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