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THE $\mathscr{L}^{p,\lambda}$ SPACES AND APPLICATIONS TO THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

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§ 1. The $\mathscr{L}^{p,\lambda}$ spaces.

In this lecture I propose to expose some results about the spaces $\mathscr{L}^{p,\lambda}$ and some of their applications to the theory of differential equations of elliptic type.

The theory of the $\mathscr{L}^{p,\lambda}$ spaces permits us to unify in a single family the spaces of Hölder continuous functions and the spaces L^p.

For some particular values of λ these spaces were already introduced some time ago by C. B. MORREY [16] and were used in the theory of differential equations of elliptic type both linear and non — linear.

Let $f(x)$ be a function defined, for simplicity on a cube Q_0 of R^n and belonging to $L^p(Q_0)$ $(p \geq 1)$. The function $f(x)$ is said to belong to the space of Morrey $L^{p,\lambda}$ if there exists a constant K such that

$$(1.1) \qquad \int_Q |fx)|^p \, dx \leq K|Q|^{1-\lambda/n}$$

for every subcube Q of Q_0 whose sides are parallel to those of Q_0.

We denote by $|Q|$ the n-dimensional measure of Q.

If $\lambda \geq 0$ one obtains a Banach space defining the norm as follows:

$$||f||^p_{L^{p,\lambda}} = \sup_{Q \subset Q_0} |Q|^{\lambda/n-1} \int_Q |f(x)|^p \, dx \, .$$

The condition that $\lambda \geq 0$ is essential because if $\lambda < 0$ then one would find that the only function belonging to $L^{p,\lambda}$ is the function 0. For $\lambda = n$ evidently we have $L^{p,n} \equiv L^p$ and for $\lambda = 0$ we have $L^{p,0} \equiv L^\infty$ for all $p \geq 1$.

More recently [13], [14], [1], [21] the spaces $\mathscr{L}^{p,\lambda}$ were introduced in the following manner: a function of $L^p(Q_0)$ is said to belong to $\mathscr{L}^{p,\lambda}$ if there exists a constant K such that

$$(1.2) \qquad \int_Q |f(x) - f_Q|^p \, dx \leq K^p |Q|^{1-\lambda/n},$$

for every subcube Q of Q_0 with sides parallel to those of Q_0, where f_Q denotes the (integral) mean value of f on Q. Let us set

$$[f]^p_{\mathscr{L}^{p,\lambda}} = \sup_{Q \subset Q_0} |Q|^{\lambda/n-1} \int_Q |f(x) - f_Q|^p \, dx \tag{1.3}$$

and

$$\|f\|_{\mathscr{L}^{p,\lambda}} = \|f\|_{L^p} + [f]_{\mathscr{L}^{p,\lambda}} . \tag{1.4}$$

In this manner $\|f\|_{\mathscr{L}^{p,\lambda}}$ will be a norm of the Banach space $\mathscr{L}^{p,\lambda}$ while $[f]_{\mathscr{L}^{p,\lambda}}$ is on the other hand a norm if we identify two functions which differ by a constant.

We observe that a function f belongs to $\mathscr{L}^{p,\lambda}$ if and only if there exists a constant K and for each subcube $Q \subset Q_0$ a constant $\bar{f}_Q$ such that

$$\int_Q |f(x) - \bar{f}_Q|^p \, dx \leq K^p |Q|^{1-\lambda/n} \tag{1.5}$$

for any subcube Q of Q_0 with sides parallel to those of Q_0. We obtain a semi-norm equivalent to $[f]_{\mathscr{L}^{p,\lambda}}$ if we take

$$\sup_{Q \subset Q_0} \inf |Q|^{\lambda/n-1} \int_Q |f(x) - \bar{f}_Q|^p \, dx$$

where the infinum is taken over all the constants $\bar{f}_Q$ associated to f and Q.

If $q \geq p$ and $\dfrac{\mu}{q} \leq \dfrac{\lambda}{p}$ then $\mathscr{L}^{q,\mu} \subset \mathscr{L}^{p,\lambda}$.

If $\lambda > 0$ the two spaces $\mathscr{L}^{p,\lambda}$ and $L^{p,\lambda}$ coincide and hence one can assume $\bar{f}_Q \equiv 0$ in (1.5). But the spaces $L^{p,0}$ and $\mathscr{L}^{p,0}$ are different. In fact, while the first coincides with the space of all (essentially) bounded functions the second coincides with a space studied by F. John and L. Nirenberg [13] which consists of functions of bounded mean oscillation and we denote this space by $\mathscr{E}_0$.

The space $\mathscr{E}_0$ consists of functions $f(x)$ for which there are two constants H and β such that

$$\text{meas}\,\{x; |f(x) - f_Q| > \sigma\} \leq H\, e^{-\beta\sigma} |Q|$$

for every subcube Q of Q_0.

This is equivalent to say that there exist two constants ϑ and K such that

$$\int_Q e^{\vartheta |f(x) - f_Q|} \, dx \leq K|Q|,$$

for every cube Q contained in Q_0.

For $p < \lambda < 0$ the space $\mathscr{L}^{p,\lambda}$ coincides with the space of Hölder continuous functions $C_{0,\alpha}$ where the exponent α is given by $\alpha = -\dfrac{\lambda}{p}$. In fact, setting

$$[f]_{0,\alpha} = \sup_{x',x'' \in Q_0} \frac{|u(x') - u(x'')|}{|x' - x''|^{\alpha}},$$

the two norms $[f]_{0,\alpha}$ and $[f]_{\mathscr{L}^{p,\lambda}}$, after identifying two functions which differ by a constant, are equivalent. This result was proved (independently) by S. CAMPANATO [1] and N. MEYERS [14].

It is important to observe that the role played by the cubes Q in the previous definitions can be substituted by any family of sets $\{E\}$ which are "regular" in the sense that for each set E of the family there exists two cubes $Q' \subset Q''$ such that

$$Q' \subset E \subset Q'', \qquad \nu^{-1} \leq \frac{|Q'|}{|Q|} \leq \nu$$

where ν is a constant independent of the particular set E considered.

Thus one can remark that the property that a function f belongs to a space $\mathscr{L}^{p,\lambda}$ is not altered by a change of variables which is bilipschitzian.

In a manner analogous to what one does in the case of the L^p spaces one can introduce also the weak $\mathscr{L}^{p,\lambda}$ spaces. A function $f(x)$ is said to belong to the space $\mathscr{L}^{p,\lambda}$ — weak if there exists a constant K such that for each cube $Q \subset Q_0$ with sides parallel to those of Q_0 we have

$$\text{meas}\left\{x \in Q;\ |f(x) - f_Q| > \sigma\right\} \leq \left(\frac{K}{\sigma}\right)^p \cdot |Q|^{1-\lambda/n}.$$

The introduction of the spaces $\mathscr{L}^{p,\lambda}$ permits us to rediscover and to generalize a classical result of C. B. MORREY.

Let $u(x) \in H^{1,p}(Q_0)$(1) and suppose that for each subcube Q of Q we have

$$\int_Q |u_x|^p \, dx \leq K^p |Q|^{1-\lambda/n}, \qquad 0 \leq \lambda \leq n,$$

with a constant K independent of Q; that is to say $u_x \in L^{p,\lambda}$. Then, if $p < \lambda$ the function u belongs to $\mathscr{L}^{\tilde{p},\lambda}$ — weak where

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{\lambda}$$

and

$$\text{meas}\,\{x \in Q;\ |u - u_Q| > \sigma\} \leq \left(\frac{K}{\sigma}\right)^{\tilde{p}} |Q|^{1-\lambda/n}.$$

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1) We denote by $H^{1,p}(\Omega)$ the completion of the functions u which together with their first derivatives are continuous in Ω with respect to the norm

$$\|u\|_{H^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_i \|u_{x_i}\|_{L^p(\Omega)}$$

while $H_0^{1,p}(\Omega)$ denotes the closure in $H^{1,p}(\Omega)$ of the functions with compact support. We will write, in the following, H^1 and H_0^1 instead of $H^{1,2}$ and $H_0^{1,2}$.

If, instead, $p = \lambda$, then $u \in \mathscr{L}^{1,0} \equiv \mathscr{E}_0$ and

$$[u]_{\mathscr{L}^{1,0}} \leq K.$$

Finally if $p > \lambda$ then $u \in \mathscr{L}^{1,\mu}$ with $\mu = \frac{\lambda}{p} - 1$; that is $u \in C_{0,\beta}$ where $\beta = 1 - \frac{\lambda}{p}$.

These results for $\lambda = n$ take a weak form of the well known Sobolev inequality.

§ 2. Interpolation in the spaces $\mathscr{L}^{p,\lambda}$.

The problem of interpolation in the spaces $\mathscr{L}^{p,\lambda}$ presents itself in an nteresting manner. To this end we shall introduce the following definitions:

Definition (2.1) — *A linear operation T on functions f defined over Q_0 is said to be of strong type $\mathscr{L}[p, (q, \mu)]$ if there exists a constant K, independent of f, such that*

$$[Tf]_{\mathscr{L}^{q,\mu}} \leq K\|f\|_{L^p}; \tag{2.1}$$

the smallest of the constants K in (2.1) *is called the strong $\mathscr{L}[p, (q, \mu)]$ norm of T.*

We now introduce the following expression:

$$\Phi_\mu(u, \sigma) = \sup_{Q \subset Q_0} [|Q|^{\mu/n-1} \operatorname{meas} \{x \in Q; |u(x) - u_Q| > \sigma\}].$$

Definition (2.2) — *A linear operation T on functions defined over Q_0 is said to be of weak type $\mathscr{L}[p, (q, \mu)]$ if there exists a constant K, independent of f, such that*

$$\Phi_\mu(Tf, \sigma) \leq \left(\frac{K\|f\|_{L^p}}{\sigma}\right)^q; \tag{2.2}$$

the smallest of the constants K in (1.5) *is called the weak $\mathscr{L}[p, (q, \mu)]$ norm of T.*

Theorem (2.1) [21] — *Let $[p_i, q_i, \mu_i]$ be real numbers satisfying the conditions*

$$p_i \geq 1, \quad p_i \leq q_i \quad (i = 1, 2); \quad p_1 \neq p_2 \quad \text{and} \quad q_1 \neq q_2.$$

For $0 < t < 1$ let $[p(t), q(t), \mu(t)]$ be defined by the relations

$$\begin{cases} \dfrac{1}{p} = \dfrac{(1-t)}{p_1} + \dfrac{t}{p_2}, \qquad \dfrac{1}{q} = \dfrac{(1-t)}{q_1} + \dfrac{t}{q_2}, \\[2ex] \dfrac{\mu}{q} = (1-t)\dfrac{\mu_1}{q_1} + t\dfrac{\mu_2}{q_2} \end{cases} \tag{2.3}$$

If T is a linear operation which is simultaneously of weak types $\mathscr{L}[p_i, (q_i, \mu_i)]$ with respective norms K_i $(i = 1, 2)$ then T is of strong type $\mathscr{L}[p, (q, \mu)]$ for $0 < t < 1$ and

$$[Tf]_{\mathscr{L}^{(q,\mu)}} \leq \mathscr{K}\, K_1^{(1-t)}\, K_2^t \|f\|_{L^p(Q_0)}$$

where $\mathscr{K}$ is a constant, independent of f, but depending on t, p_i, q_i, μ_i and it is bounded for t away from 0 and 1.

An useful corollary of theorem (2.1) is the following.

Corollary (2.1) — *Any time a linear operation T maps L^{p_1} into a space of Hölder continuous functions and L^{p_2} into a (weak) L^{q_2} — space, then exist there a special $\bar{p}$ such that T maps $L^{\bar{p}}$ into the space $\mathscr{E}_0$.*

For generalizations of this theorem see [8], [9], [18].

Theorem (2.2) [5] — *Let $[p_i, (q_i, \mu_i)]$ be real numbers such that p_i, $q_i \geq 1$ $(i = 1, 2)$. If T is a linear operation (in general on complex valued function on Q_0) which is simultaneously of strong types $\mathscr{L}[p_i, (q_i, \mu_i)]$ with respective norms K_i $(i = 1, 2)$ then T is of strong type $\mathscr{L}[p, (q, \mu)]$ where p, q, μ are defined for $0 \leq t \leq 1$ by (1.6) and further the following estimate holds*

$$[u]_{\mathscr{L}^{q,\mu}} \leq K_1^{(1-t)} K_2^t \|u\|_{L^p}.$$

The previous theorems generalize respectively the theorems of interpolation of Marcinkiewicz and of Riesz—Thorin.

Another theorem of interpolation is found to be very useful; it completes the theorems above. For this purpose we shall introduce the spaces N^p.

We shall denote by $\bar{S}$ the family of systems S of a finite number of subcubes Q_i no two of which have an interior point in common and having their sides parallel to those of Q_0 $(\bigcup_i Q_i = Q_0)$.

For any (real or complex valued) function $u \in L^1(Q_0)$ and for any $1 < p < +\infty$ we consider the expressions of the form

$$\sum_i \left| \int_{Q_i} |u - u_{Q_i}| \, dx \right|^p |Q_i|^{(1-p)}$$

where Q_i runs through a system $S \in \bar{S}$.

For $1 < p < +\infty$ set

$$[u]_{N^p} = \sup_{\{Q_i\} \equiv S \in \bar{S}} \left\{ \sum_i \left| \int_{Q_i} |u - u_{Q_i}| \, dx \right|^p |Q_i|^{(1-p)} \right\}^{1/p}$$

and the following.

Definition (2.3) — *A function u is said to belong to N^p $1 \leq p < +\infty$ if $[u]_{N^p} < +\infty$. We observe that $[u]_{N^p}$ defines a semi-norm in N^p and we obtain a Banach space by taking*

$$||u||_{N^p} = ||u||_{L^1} + [u]_{N^p}$$

as the norm in N^p.

If $q \geq p$, *then* $N^q \subset N^p$.

If $u \in L^1(Q_0)$ *then we have*

$$\lim_{p \to +\infty} [u]_{N^p} = [u]_{\mathscr{L}^{1,0}} = \mathscr{E}_0$$

i.e. *we may set* $N^\infty = \mathscr{L}^{(1,0)} = \mathscr{E}_0$.

In connection with these spaces N^p the following result due to F. John and L. Nirenberg holds [13].

If $u \in N^p$ *with* $p > 1$ *then there exists a constant* C *such that, for any cube* $Q \subset Q_0$, *we have*

$$\text{meas}\,\{x \in Q;\, |u(x) - u_Q| > \sigma\} \leq C \left(\frac{[u]_{N^p(Q)}}{\sigma}\right)^p.$$

Conversely, one can show that if u is a measurable function satisfying the condition

$$\text{meas}\,\{x \in Q;\, |u(x) - u_Q| > \sigma\} \leq C \left(\frac{K(Q)}{\sigma}\right)^p$$

for each cube $Q \subset Q_0$ where $K(Q)$ are constants with the following property: for any system $\{Q_i\} \equiv S \in \bar{S}$, introduced above, and for some $r \leq p$ we have

$$\sum_i |K(Q_i)|^r \leq |K(Q)|^r,$$

then $u \in N^p$ and we have

$$[u]_{N^p} \leq \frac{2}{(p-1)^{1/p}} K.$$

In fact. we have

$$\int_Q |u(x) - u_Q|\, dx \leq \frac{2K(Q)}{(p-1)^{1/p}} |Q|^{1-1/p}$$

from which it follows that for $\{Q_i\} \equiv S \in \bar{S}$,

$$\sum |Q_i|^{1-p} \left| \int_{Q_i} |u(x) - u_{Q_i}|\, dx \right|^r \leq \frac{2^p}{p-1} |K(Q_i)|^r\, |K(Q_i)|^{p-r} \leq \frac{2^p}{p-1} |K(Q)|^p.$$

Admitting this result we have the following theorem of interpolation.

Theorem (2.3) [22] — *Let* T *be a linear operation defined on the class* $\mathscr{F}$ *of (real valued) simple functions on* Q_0 *such that*

$$[Tu]_{\mathscr{L}^{(1,0)}} \leq K_1||u||_{L^{p_1}},$$

$$[Tu]_{N^{q_2}} \leq K_2||u||_{L^{p_2}}$$

where $p_1, p_2, q_2 > 1$ *with* $q_2 \geq p_2$. *If* $p, q \geq 1$ *are defined by*

$$\frac{1}{p} = \frac{(1-t)}{p_1} + \frac{t}{p_2}, \qquad \frac{1}{q} = \frac{t}{q_2} \tag{2.4}$$

then

$$||Tu - (Tu)_{Q_0}||_{L^q} \leq \mathscr{K} K_1^{(1-t)} K_2^t ||u||_{L^p} \quad \textit{for} \quad u \in \mathscr{F}$$

where $\mathscr{K}$ *is a constant which is bounded if* t *is away from* 0 *and* 1.

The theorem is valid also for $p_1 = +\infty$.

Before giving some applications of this theorem we observe that if $f \in L^p$ — weak and

$$\text{meas}\,\{x \in Q;\ |f(x)| > \sigma\} \leq \left(\frac{K(Q)}{\sigma}\right)^p$$

and if there exists an $r < p$ such that $\sum |K(Q_i)|^r \leq |K(Q)|^r$, then

$$[f]_{N^p} \leq \text{const}\ |K(Q)|.$$

In fact, then there exists a constant $C(p)$ such that

$$\text{meas}\,\{x \in Q;\ |f(x) - f_Q| > \sigma\} \leq C(p) \left(\frac{K(Q)}{\sigma}\right)^p.$$

In particular, the assumption is satisfied provided $f \in L^p$ with $K(Q) = \int_Q |f|^p\, dx$.

We deduce from theorem (2.3) the following results:

Theorem (2.4) — *Let* T *be a linear operation defined on the class* $\mathscr{F}$ *of simple functions on* Q_0 *such that*

$$[Tu]_{\mathscr{L}^{1,0}} \leq K_1||u||_{L^{p_1}}; \qquad ||Tu||_{L^{q_2}} \leq K_2||u||_{L^{p_2}},$$

where $p_1, p_2, q_2 > 1$ *with* $q_2 \geq p_2$. *Then*

$$||Tu||_{L^q} \leq \mathscr{K}\, K_1^{(1-t)} K_2^t ||u||_{L^p},$$

where $\mathscr{K}$ *is a constant which is bounded if* t *is away from* 0 *and* 1 *and* p *and* q *are given by* (2.4).

The theorem is valid also for $p_1 = +\infty$.

Theorem (2.4) can be extended in the following way

Theorem (2.5) — *Let* T *be a linear operation defined on the class* $\mathscr{F}$ *of simple functions on* Q_0 *such that*

$$[Tu]_{\mathscr{L}^{1,0}} \leq K_1||u||_{L^{p_1}}$$

$$\text{meas}\ \{|Tu| > \sigma\} \leq \left(\frac{K_2||u||_{L^{p_2}}}{\sigma}\right)^{q_2}$$

where $p_1 \geq 1$, $p_2 \geq 1$, $q_2 > 1$. *Then*

$$||Tu||_{L^q} \leq \mathscr{K}\, K_1^{1-t} \,.\, K_2||u||_{L^p}$$

where $\mathscr{K}$ *is a constant which is bounded if* t *is away from* 0 *and* 1 *and* p *and* q *are given by* (2.3).

The theorem holds also for $p_1 = +\infty$.

We are going to sketch the proof of this theorem making use of a trick introduced by CAMPANATO in giving a new proof of theorem (2.4) [4].

Let S a fixed system of a finite number of subcubes Q_i no two of which have an interior point in common and having their sides parallel to those of Q_0. Set

$$\mathscr{T}(u) = \frac{1}{|Q_i|} \int_{Q_i} |Tu - (Tu)_{Q_i}|\, dx \quad \text{in} \quad Q_i.$$

The map $\mathscr{T}(u)$ is sub-linear and satisfy

$$||\mathscr{T}(u)||_{L^\infty} \leq K_1||u||_{L^{p_1}}$$

$$\text{meas}\ \{|\mathscr{T}(u)| > \sigma\} \leq \left(\frac{K_2'||u||_{L^{p_2}}}{\sigma}\right)^{q_2}.$$

The first inequality is obvious; the second one can be proved easily. In fact if we denote by Q_i' the cubes of S for which one has

$$\int_{Q_i'} |Tu - (Tu)_{Q_i'}|\, dx > \sigma|Q_i'|\,,$$

it follows

$$\sigma \sum |Q_i'| \leq 2 \int_{\cup Q_i'} |Tu|\, dx \leq 2\left(1 - \frac{1}{q_2 - 1}\right) K_2||u||_{L^{p_2}} \left(\sum |Q_i'|\right)^{1-1/q_2},$$

and then

$$\text{meas}\ \{|\mathscr{T}(u)| > \sigma\} = \sum |Q_i'| \leq \left\{2\left(1 - \frac{1}{q_2 - 1}\right) K_2||u||_{L^{p_2}}/\sigma\right\}^{q_2}.$$

Applying the theorem of MARCINKIEWICZ it follows that

$$||\mathscr{T}(u)||_{L^q} \leq \mathscr{K}\, K_1^{1-t}\, K_2^t||u||_{L^p}$$

where p and q are given by (2.3) and $\mathscr{K}$ is a constant which is bounded if t stay away from 0 and 1.

But, from the definition of $\mathscr{T}(u)$, we have

$$\{\sum_i \mid \int_{Q_i} |Tu - (Tu)_{Q_i}|\, dx|^q\, |Q_i|^{1-q}\}^{1/q} \leq \mathscr{K}\, K_1^{1-t} \,.\, K_2^t||u||_{L^p},$$

and thus, since S is arbitrary

$$[Tu]_{N^q} \leq \mathscr{K} K_1^{1-t} K_2^t ||u||_{L^p}$$

therefore, applying the lemma of F. JOHN and L. NIRENBERG,

$$\text{meas}\, \{|Tu - (Tu)_Q| > \sigma\} \leq \left(\frac{\mathscr{K}' K_1^{1-t} K_2^t ||u||_{L^p}}{\sigma} \right)^q .$$

Then making use again of the theorem of MARCINKIEWICZ one has

$$||Tu - (Tu)_Q||_{L^q} \leq \mathscr{K}'' \,.\, K_1^{1-t} \,.\, K_2^t ||u||_{L^p}$$

and from this the conclusion of the theorem follows easily.

It would be interesting to know whether the theorem (2.5) holds for $q_2 = 1$.

Theorem (2.5) can be considered as a generalization of the theorem of MARCINKIEWICZ where the space $\mathscr{E}_0$ replaces usefully the space L^∞.

From the corollary (2.1) and theorem (2.5) the theorem of interpolation follows:

Theorem (2.6) — *Let T be a linear mapping such that, continuously*

$$T \;:\; L^{p_1} \to C^{0,\alpha}$$

$$T \;:\; L^{p_2} \to L^{q_2} \qquad (\textit{weak}),\ q_2 > 1,\ p_2 \leq q_2$$

then, for $\dfrac{1}{p} = \dfrac{1-t}{p_1} + \dfrac{t}{p_2}$, $0 < t < 1$, *set* $\vartheta = \alpha \Big/ \left(\alpha + \dfrac{n}{q_2}\right)$

$$T : L^p \to \begin{cases} C^{0,\beta}, & \textit{for} \quad 0 \leq t < \vartheta, \qquad \beta = (1-t)\,\alpha - \dfrac{n}{q_2}\, t \\ \mathscr{E}_0, & \textit{for} \quad t = \vartheta \\ L^q, & \textit{for} \quad \vartheta < t < 1, \qquad \dfrac{1}{q} = \dfrac{1}{q_2} \left\{ \left(1 + \dfrac{\alpha q_2}{n}\right) t - \dfrac{\alpha q_2}{n} \right\} \end{cases}$$

The previous results on interpolation show that the $\mathscr{L}^{p,\lambda}$ spaces form a family of spaces of interpolation with respect to special families of spaces, the L^p — spaces. There might be more general families of spaces than the L^p spaces with respect to which the spaces $\mathscr{L}^{p,\lambda}$ are spaces of interpolation (see [19]), but, on the other side, the spaces $\mathscr{L}^{p,\lambda}$ are not spaces of interpolation with respect to the family of the spaces $\mathscr{L}^{p,\lambda}$ themselves. E. M. STEIN and A. ZYGMUND [24] have indeed proved this fact adapting an example given by HARDY and LITTLEWOOD [11]. They have proved that there exists a linear mapping T which maps continuously $C^{0,\alpha}$ into $C^{0,\alpha}$, L^2 into L^2 but it does not map $\mathscr{E}_0$ into $\mathscr{E}_0$.

Thus, it is interesting to find families of operations which leave the spaces $\mathscr{L}^{p,\lambda}$ invariant. One of these families of operators has been found by J. PEETRE [17].

This family includes the singular integral transform of CALDERON—ZYGMUND.

A consequence of theorem (2.4) is the following.

Theorem (2.6) — *If the operator T leaves the spaces $\mathscr{L}^{p,\lambda}$ invariant for a fixed p and for $0 \leq \lambda < n$, then T leaves invariant the spaces L^q for all $q \geq p$.*

In fact, one has

$$T : L^\infty \to \mathscr{E}_0$$
$$T : L^p \to L^p$$

and, thus, from theorem (2.4), follows

$$T: \; L^q \to L^q \qquad \text{for} \qquad q \geq p.$$

Making use of the interpolation theorem (2.4) it is possible to give an easy proof of a theorem by HORMANDER [12], (see [23], [19]).

Consider the translation invariant mapping

$$Tf = \int K(x-y) f(y)\, \mathrm{d}y$$

and assume that the Fourier transform $\hat{K}$ of K, as distribution, satisfies: $|\hat{K}(x)| \leq A$. Moreover assume that

$$\int_{|x| \geq 2|y|} |K(x-y) - K(x)|\, \mathrm{d}x \leq A.$$

Then Tf maps L^2 into L^2 because of the first assumption. It can be proved that T maps L^∞ into $\mathscr{E}_0$ [23], [19].

It follows, from theorem (2.4) that Tf maps L^p into L^p for $p \geq 2$.

By a duality argument the same conclusion holds for $p > 1$.

The proof that T maps L^∞ into $\mathscr{E}_0$ is easy and we are going to sketch it here.

Let f be a bounded function ($|f(x)| \leq 1$) and write $u(x) = Tf$. Fix a cube Q, which we may assume centered at the origin. Let us split $f = f_1 + f_2$ where $f_1(x) = f(x)$ in the sphere S' of diameter twice that of Q and having the same center that Q; $f_1(x) = 0$ outside this sphere. Write $u_i(x) = T(f_i)$ ($i = 1,2$); $u(x) = u_1(x) + u_2(x)$.

Now

$$\int_Q |u_1(x)|^2\, \mathrm{d}x \leq A^2 \int_S |f_1(x)|^2\, \mathrm{d}x \leq A^2 c|Q|.$$

Next

$$u_2(x) = \int K(x-y) f_2(y)\, \mathrm{d}y.$$

Let

$$u_Q = \int K(y) f_2(y)\, \mathrm{d}y.$$

Therefore

$$|u_2(x) - u_Q| \leq \int_{y \notin S} |K(x-y) - K(y)| \leq A.$$

Combining the informations above we get

$$\frac{1}{|Q|}\int_Q |u(x)-u_Q|^2\,dx \le A^2(1+c)$$

i.e.: $u \in \mathscr{E}_0$.

§ 3. Application to the theory of differential equations.

C. B. Morrey has extensively used the spaces $\mathscr{L}^{2,\lambda}$ for $0 < \lambda < n$ in the theory of differential equations of elliptic type linear and non-linear [16]. Some of his results can be extended making use of the spaces $\mathscr{L}^{2,\lambda}$ either for positive or negative values of λ. We mention the following theorem which generalizes a theorem by Morrey [15]. It can be proved essentially in the same way.

Let $a_{ij}(x)$ $(i, j = 1, 2, \ldots, n)$ be bounded measurable functions in an open set Ω, satisfying

$$\sum_{i,j}^{1\ldots n} a_{ij}(x)\,\xi_i\xi_j \ge \nu(\xi)^2 \qquad \nu = \text{const} > 0, \qquad \xi \in R^n$$

and let f_i be n functions of $L^2(\Omega)$. Let u be a function of $H^1(\Omega)$ which, with the usual convention on the sum, satisfies

$$(3.1)\qquad \int_\Omega a_{ij}(x)\,u_{x_i}v_{x_j}\,dx = \int_\Omega f_i v_{x_i}\,dx \qquad \text{for all} \qquad v \in H_0^1(\Omega).$$

The following theorem holds

Theorem (3.1) — *There exists a constant* λ_0, $0 < \lambda_0 < 2$ *such that, for* $f_i \in \mathscr{L}^{2,\lambda}$ *with* $\lambda_0 < \lambda \le n$, *one has, in any* Ω' *with* $\overline{\Omega}' \subset \Omega$, $u_{x_i} \in L^{2,\lambda}$ *and, consequently* $u \in \mathscr{L}^{\tilde{2},\lambda} \subset \mathscr{L}^{2,\lambda-2}$ *where* $\frac{1}{\tilde{q}} = \frac{1}{2} - \frac{1}{\lambda}$ *for* $\lambda > 2$, *and* $u \in \mathscr{L}^{2,\lambda-2}$ *for* $\lambda \le 2$.

In [15] this theorem is proved assuming $\lambda_0 < \lambda < 2$; with such a limitation the function u is Hölder continuous.

From theorem (3.1) and using the interpolation theorem (2.4) it is possible to deduce some estimates found in [20]:

If $f_i \in L^p$, $p > 2$, *then* (*i*) $u \in L^{p^*}$ *where* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ *for* $p < n$ (*ii*) $u \in \mathscr{E}_0$ *for* $p = n$, (*iii*) u *is Hölder continuous for* $p > n$.

When in (3.1) the coefficients $a_{ij}(x)$ are assumed to be Hölder continuous more informations can be obtained for u.

Campanato [2] has proved the following theorem.

Theorem (3.2) — *Let f_i be in $\mathscr{L}^{2,\lambda}$, with $-2 < \lambda \leq n$, and let Ω' be a open set such that $\overline{\Omega}' \subset \Omega$.*

(*i*) *If the coefficients a_{ij} are continuous and $0 < \lambda \leq n$, then, $u_{x_i} \in \mathscr{L}^{2,\lambda}$ in Ω'.*
(*ii*) *If a_{ij} are Hölder continuous in $\overline{\Omega}$ and $\lambda = 0$ then, in Ω', $u_{x_i} \in \mathscr{E}_0$.*
(*iii*) *If $a_{ij} \in C^{0,-\lambda/2}$ and $-2 < \lambda < 0$ then $u_{x_i} \in \mathscr{L}^{2,\lambda} \equiv C^{0,-\lambda/2}$.*
If Ω is "smooth" and $u \in H_0^1(\Omega)$, then the same conclusions hold in $\overline{\Omega}$.

This theorem unifies Caccioppoli—Schauder estimates with Morrey's estimates.

The proof of this theorem does not make use of the potential theory.

From theorem (3.2) and the interpolation theorem (2.4) it follows that when $f_i \in L^p(\Omega)$, $p > 1$ one has $u_{x_i} \in L^p(\Omega)$. This method has been used in [6].

It should be mentioned that a generalization of the spaces $\mathscr{L}^{p,\lambda}$, with respect to a different norm in R^n, has been considered. This generalization turns out to be useful in dealing with parabolic and quasi elliptic differential equations. See [7], [3], [10].

BIBLIOGRAPHY

[1] S. Campanato — *Proprietà di holderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa, Vol. 17, 1963, pp. 175 — 188.

[2] S. Campanato — *Equazioni ellittiche del II° ordine e spazi $\mathscr{L}^{(2,\vartheta)}$*, Ann di. Matematica, Vol. 69, 1965, pp. 321—382.

[3] S. Campanato — *Equazioni paraboliche del secondo ordine e spazi $\mathscr{L}^{2,\vartheta}(\Omega, \delta)$*, Ann .di Matematica, Vol. 73, 1966, pp. 55—102.

[4] S. Campanato — *Su un teorema di G. Stampacchia*, Ann. Sc. Norm. Sup. Pisa, Vol. XX, 1966, pp. 649—652.

[5] S. Campanato and M. K. V. Murthy — *Una generalizzazione del teorema di Riesz-Thorin*, Ann. Sc. Norm. Sup. Pisa, Vol. 19, 1965, p. 87—100.

[6] S. Campanato and G. Stampacchia — *Sulle maggiorazioni in L^p nella teoria dell' equazioni ellittiche*, Boll. U. M. I., Vol, XX, 1965, pp. 393—399.

[7] G. da Prato — *Spazi $\mathscr{L}^{(p,\theta)}(\Omega, \delta)$ e loro proprietà*, Ann. di Matem. Vol. 69, 1965, pp. 383—392.

[8] P. Grisvard — *Semi-groups faiblement continus et interpolation*, C. R. Acad. Sc. Paris, Vol. 259, 1964, pp. 27—29.

[9] P. Grisvard — *Commutativité de deux foncteurs d'interpolation et applications*, Thèse.

[10] E. Giusti — *Equazioni quasi-ellittiche e spazi $\mathscr{L}^{(p,\vartheta)}(\Omega, \delta)$*, (I) to appear in Ann. di Matem., (II) Ann. Sc. Norm. Sup.

[11] G. H. Hardy and J. E. Littlewood — *Theorems concerning the mean values of analytic or harmonic functions*, Quart. J. of Math. Vol. 12, 1941, pp. 221—256.

[12] L. Hörmander — *Estimates for translation invariant operators in L_p spaces*, Acta *Math., 104 (1960), pp. 93—140.*

[13] F. John and L. Nirenberg — *On functions of bounded mean oscillations,* Comm. Pure Appl. Math., Vol. 14, 1961, pp- 415—426.

[14] G. N. Meyers — *Mean oscillation over cubes and Hölder continuity,* Proc. Amer. Math. Soc. Vol. 15, 1964, p. 717—721.

[15] C. B. Morrey — *Second order elliptic equations in several variables and Hölder continuity,* Math. Z. *72* (1959), 146—164.

[16] C. B. Morrey — *On the solution of quasi-linear elliptic partial differential equations,* Trans. Am. Math. Soc. Vol. 43, 1938, pp. 126—166.

[17] J. Peetre — *On convolution operators leaving $\mathscr{L}^{p,\lambda}$ spaces invariant,* Ann. Matem. Vol. 72, 1966, pp. 295—304.

[18] J. Peetre — *Espaces d'interpolation et théorème de Soboleff,* Ann. Inst. Fourier, Grenoble 16, 1966, pp. 279—317.

[19] S. Spanne — *Sur l'interpolation entre les espaces $\mathscr{L}_k^{p,\Phi}$*. Ann. Sc. Norm. Sup. Pisa, Vol. 20, 1966, pp. 625—648.

[20] G. Stampacchia — *Régularisation des solutions de problèmes aux limites elliptiques à données discontinues,* Intern. Symp. on Lin. Spaces, Jerusalem, 1960

[21] G. Stampacchia — *$\mathscr{L}^{(p,\lambda)}$ spaces and interpolation,* Comm. Pure and Appl. Math. Vol. 17, 1964, pp. 293—306.

[22] G. Stampacchia — *The Spaces $\mathscr{L}^{(p,\lambda)}$, $N^{(p,\lambda)}$ and interpolation,* Ann. Sc. Norm. Sup. Pisa. Vol. 19, 1965, pp. 443—462.

[23] E. M. Stein — *Singular integrals, harmonic functions, and differentiability properties of functions of several variables,* to appear.

[24] E. M. Stein and A. Zygmund — *Boundedness of translation invariant operators on Hölder spaces and L^p spaces,* to appear.

[25] A. Zygmund — *Trigonometric Series,* Cambridge University Press, 1959.