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ON THE LINEAR AND QUASILINEAR PARABOLIC EQUATIONS

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1. If the coefficients in equations are smooth then it take place the unique solvability and the strong estimates for solutions of large classes of parabolic systems with very general boundary conditions. The first estimate of the same type was the inequality.

$$(1)\qquad \int_{Q_T} (u_t^2 + u_{xx}^2)\,\mathrm{d}x\,\mathrm{d}t + \max_{0 \leq t \leq T} \int_\Omega u^2(x, t)\,\mathrm{d}x \leq$$
$$\leq c \int_{Q_T} (u_t - a_{ij}(x, t)\, u_{x_ix_j} + \dots)^2\,\mathrm{d}x\,\mathrm{d}t + c_1 \int_\Omega u^2(x, 0)\,\mathrm{d}x \equiv$$
$$\equiv c \int_{Q_T} (Zu)^2\,\mathrm{d}x\,\mathrm{d}t + c_1 \int_\Omega u^2(x, 0)\,\mathrm{d}x,$$
$$Q_T = \{(x, t) : x \in \Omega, t \in [0, T]\},$$

which is hold for any arbitrary function $u(x, t)$, satisfying one of classical homogenous boundary conditions. After that had been proved the strong estimates for the parabolic operator in the space $L_p(Q_T)$, $p > 1$. (SOLONNIKOV) and in the HÖLDER space (A. FRIEDMAN). These estimates were generalised for parabolic systemes. The most wide class of systemes and boundary conditions was considered by SOLONNIKOV. The unique solvability had been proved with these estimates together.

The kernel of all these estimates [exepte my prove of the inequality (1)] is provided by the Korn—Schauder "gluiing" idea, which permits one to reduce the general estimation problem to some canonical estimation problems for systems with constant coefficients. For this approach to be feasible, however it is necessary that the coefficients of the leading termes of system be continuous.

2. Investigations of parabolic equations and systems with discontinuous coefficients have bean based principally on the energy inequality (see the papers LADYZENSKAYA, VISHIK, LIONS, BROWDER and others). This inequality holds for a harrower class of systems than that mentioned above — for so-called strongly-parabolic systems, under some simplest boundary

conditions. This class of parabolic systems belong the parabolic equations of the form

(2) $$u_t - Z^{(2m)}u = f,$$

where $Z^{(2m)}u$ is an elliptic operator of the order $2m$ with principal part in divergence form: $Z^{(2m)}u = D_x^m(a(x, t)\, D_x^m u)\, t$.

The energy inequality enables one to prove the existence of generalised solutions, which for the equations (2) have derivatives $D_x^m u$ belonging to $L_2(Q_T)$ and which are continuous with respect to t in the norm of $L_2(\Omega)$ (we require that $u_0 = u(x, 0) \in L_2(\Omega)$, the inhomogeneous term f may be of rather general nature). But it gives no additional information about the solution even when u_0 and f are very smooth.

It was not possible on the basis of the methods available up to 1956—57 to derive any conclusions about the improvements of the differentiability properties of the solutions of parabolic equations with discontinuous coefficients when u_0 and f (but not coefficients) become smoother. This situation took place even for equations of the second order.

Some ten years ago, however, new methods began to develop following upon the pioneering work of Nash and De Georgi. By means of these methods had been established a series of new principles (relations) for linear equations of second order and they proved to be usfull for the study of quasilinear equations as well.

In joint papers by URALTZEVA and myself were investigated equations of the form

(3) $$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x, t)\, u_{x_j} + a_i(x, t)\, u) + b_i u_{x_i} + au = f + \frac{\partial f_i}{\partial x_i}$$

under the conditions that the coefficients a_{ij} be arbitrary measureable functions satisfying the inequality

(4) $$\nu\xi^2 \leq a_{ij}\xi_i\xi_j \leq \mu\xi^2, \qquad \nu, \mu = \text{const} > 0,$$

and that the coefficients of the lower terms, as well as inhomogeneous teims f and f_i in (3), belong to the spaces $L_{q_k,\infty}(Q_T)$ with appropriate q_k (the norm in $L_{q_k,\infty}(Q_T)$ is defined by $||u||_{q,\infty,Q_T} = \operatorname{vrai\,max}_{0 \leq t \leq T} (\int_\Omega |u|^q\, dx)^{\frac{1}{q}}$.

Some examples were also given demonstrating that the dependence of the degree of smoothness of solutions of equation (3) on the parameters q_k established in the above mentioned joint papers is best possible.

Slight modifications of the methods developed by stadying these equations (3) and of the corresponding elliptic equations of the second order enable one to investigate the cases in which the coefficients and inhomogeneous terms are elements of $L_{q_k,r_k}(Q_T)$. [The norm in $L_{q,r}(Q_T)$ being defined by

$\|u\|_{q,r,Q_T} = \{\int_0^T (\int_\Omega |u|^q \,dx)^{\frac{r}{q}} \,dt\}^{\frac{1}{r}}]$. In a joint paper [*] by N. N. URALTZEVA, A. V. IVANOV, A. L. TRESKUNOV and myself the study was made of the ranges of q_k and r_k in which solutions u of equation (3):

a) belong to the spaces $L_{q,r}$

b) give a finite value to the integral $\int \exp\{\lambda u(x,t)\}\,dx\,dt,\ \lambda > 0$;

c) have a bounded vraimaximum $|u|$;

d) have a finite HÖLDER norm $|u|_{x,t}^{(\alpha,\frac{\alpha}{2})}$.

The above has all been carried out for generalised solutions of equation (3) belonging to the space $V_2^{1,0}(Q_T)$. This space is the completion in the norm

$$(5)\qquad |u|_{Q_T} = \max_{0\le t\le T} \left(\int_\Omega u^2(x,t)\,dx\right)^{\frac{1}{2}} + \left(\int_{Q_T} |u_k|^2\,dx\,dt\right)^{\frac{1}{2}}.$$

of the set of smooth functions. Examples were constructed showing that the regularity conditions given in [*] are exact (in the sence that the indices q_k, r_k cannot be reduced). I remark also that as it often takes place passage from the spaces $L_{q,\infty}$ to the full scale of spaces $L_{q,r}$ enable us to make the results obtained more transparent (observeable) and more complete.

As an example of the results obtained in [*] I shall formulate the following theorem:

Theorem I. *Assume that for equation* (3) *the inequality* (4) *and the conditions*

$$(6)\qquad \sum_{i=1}^n a_i^2,\qquad \sum_{i=1}^n b_i^2,\qquad a \in L_{q,r}(Q_T),\qquad \frac{1}{r}+\frac{n}{2q}\le 1,$$

hold. Then if

$$f_i \in L_2(Q_T),\qquad f \in L_{q,r_1}(Q_T),\qquad \frac{1}{r_1}+\frac{n}{2q_2}\le 1+\frac{n}{4},$$

$$u|_{t=0} = u_0(x) \in L_2(\Omega)\qquad \text{and } u|_S = 0$$

the first initial boundary value problem for equation (3) *is uniquely solvable in the space* $V_2^{1,0}(Q_T)$. *(To be more precise, uniquely solvable in* $V_2^{1,\frac{1}{2}}(Q_T)$*).*

If f and f_i satisfy the more restrictive conditions

$$(7)\qquad \begin{aligned} &f \in L_{q_2,r_2}(Q_T), && \frac{1}{r_2}+\frac{n}{2q_2}\le 1+\frac{n}{4}\Theta,\\ &\sum_{i=1}^n f_i^2 \in L_{q_3,r_3}(Q_T), && \frac{1}{r_3}+\frac{n}{2q_3}\le 1+\frac{n}{2}\Theta, \end{aligned}$$

where $\Theta \in (0,1)$, then the solution has the sharper properties:

$$u \in L_{\hat q,\hat r},\qquad \frac{1}{\hat r}+\frac{n}{2\hat q}=\frac{n}{4}\Theta.$$

(Remark: *from the assumption* $u \in V_2^{1,0}(Q_T)$ *one can only conclude that* $u \in L_{q,r}(Q_T)$, *where* $\frac{1}{r}+\frac{n}{2q}=\frac{n}{4}$). *If the constants* q *and* r *in* (6) *satisfy the inequality*

$$\frac{1}{r}+\frac{n}{2q}<1 \tag{8}$$

and the parameter Θ *in* (7) *vanishes, then* $\int_{Q_T} \exp\{\lambda u(x, t)\}\, dx\, dt$ *must be finite for some* $\lambda > 0$. *If* $\sum_{i=1}^{n} a_i^2$, $\sum_{i=1}^{n} b_i^2$, a, $\sum_{i=1}^{n} f_i^2$, $f \in L_{q,r}(Q_T)$ *and* q, r *satisfy* (8) *then* u *is Hölder-continuous in* (x, t).

In a particular case, when $a_i \equiv f_i \equiv f \equiv 0$ *and* $a(x, t) \geq 0$, *for weak solutions* $u(x, t)$ *takes place the maximum principle, that is:*

$$\min\{0; \operatorname{vraimin}_{\Gamma_T} u\} \leq u(x, t) \leq \max\{0; \operatorname{vraimax}_{\Gamma_T} u\}.$$

If $a(x, t) \equiv 0$, *then* $\operatorname{vraimin}_{\Gamma_T} u \leq u(x, t) \leq \operatorname{vraimax}_{\Gamma_T} u$.

In formulating theorem I have not indicated the allowable ranges of variation of q and r — these depend on the dimension n in (6).

3. Now I should like to give an example demonstrating the necessity of the restrictions of type (6) imposed on the degree of singularity of coefficients in (3). The function $u = e^{\frac{-|x|^2}{4t}}$ which vanishes when $t = 0$ presents the solution of the Cauchy problem in half space $\{(x, t): x \in E_n, t \geq 0\}$ for the equation

$$u_t - \Delta u + n \sum_{i=1}^{n} \frac{x_i}{|x|^2} u_{x_i} = 0, \tag{9}$$

and for the equation

$$u_t - \Delta u - \frac{n}{4t} u = 0 \tag{10}$$

as well.

This solution evidently belongs to $V_2^{1,0}$. Moreover by smoothing u with respect to t or x one may construct almost-classical solutions of (9) and (10). This shows that if one of the most important properties of parabolic initial value problems, namely their deterministic nature, is to be retained, singularities of the coefficients $b_i(x, t)$ of order $\frac{1}{|x|}$ and singularities of the coeficients $a(x, t)$ of order $\frac{1}{t}$ must be excluded. Conditions like (6) take care of it.

4. All the above-mentioned relations have been established only for single parabolic equations and for certain limited classes of parabolic systems of second order. In the heart of these considerations lies the maximum principle (albeit in a disquised and unusual form). It would be interesting to know whether similar relationships hold for equations of higher order.

All the results mentioned above apply only to the equation with principal part of divergent form. For the equation

$$u_t - Mu \equiv u_t - a_{ij}u_{x_ix_j} + a_iu_{x_i} + au = f \tag{11}$$

of non-divergence form with arbitrary discontinuous bounded measurable coefficients $a_{ij}(x, t)$ in more than 2 dimensions the information available is extremly scarce. In a book by Uraltzeva and myself are the examples showing the pathological properties of the equation (11) in which the operator M has the form

$$M_0u = a_{ij}u_{x_ix_j}, \qquad a_{ij} = \delta_i^j + \mu \frac{x_ix_j}{|x|^2}. \tag{12}$$

Another such pathological property is the following: the operator M_0 does not admit the closure in $L_2(\Omega)$ if $n > 2$. In fact, when $\mu = \dfrac{n}{n-2}$ for the sequence $u_{\varepsilon,\eta}(x) = (|x|^2 + \eta)^{1-\frac{n}{4}+\varepsilon} - (|x|^2 + \eta)^{1-\frac{n}{4}+\frac{\varepsilon}{2}}$ of functions we have

$$\frac{||u_{\varepsilon,\eta}||_{L_2}(|x| \leq 1)}{||M_0u_{\varepsilon,\eta}||_{L_2}(|x| \leq 1)} \to 0 \text{ as } \varepsilon \to 0 \text{ and } \eta = \eta(\varepsilon) \to 0.$$

5. In the joint work Uraltzeva and myself quasilinear equations of the form

$$u_t - \frac{\partial}{\partial x_i} a_i(x, t, u, u_x) + a(x, t, u, u_x) = 0, \tag{13}$$

$$u_t - a_{ij}(x, t, u, u_x)\, u_{x_ix_j} + a(x, t, u, u_x) = 0, \tag{14}$$

have also been considered. For equations (13) generalised and classical solutions were both studied. For equations (14) principally classical solutions were studied.

We have analysed the smoothness properties of the full set of solutions of the above equations and the unique solvability „in the large" in the spaces of smooth functions of the classical boundary value problems for these equations. In these investigations we assume that the functions $a_i(x, t, u, p)$, $a(x, t, u, p)$ and $a_{ij}(x, t, u, p)$ are smooth with respect to u and p and that as functions of x and t they belong to the spaces $L_{q,r}$. I shall not give a detailed account of these results, as they together with the above-mentioned results on linear parabolic equations form the principal part of the forthcoming book on parabolic equations by Uraltzeva, Solonnikov and myself which should be appeared by the end of the present year. To indicate the nature of these results however I will state one result concerning the general class (14) of parabolic equations:

Theorem 2. *Let u be an arbitrary generalised solutions of the equation* (14) *belonging to class M, i.e. suppose that u is bounded, has generalised derivatives u_t and u_{xx} belonging to $L_2(Q_T)$, that the derivatives u_x are bounded and depend continuosly on t in the norm of $L_2(\Omega)$ and that u satisfies equation* (14) *almost everywhere.*

Suppose that the functions $a_{ij}(x, t, u, p)$ are differentiable with respect to x, u and p in a neighborhood of the surface $u = u(x, t)$, $p = u_x(x, t)$ aud that on this surface they satisfy the conditions

$$\nu\xi^2 \leq a_{ij}(x, t, u, p)\,\xi_i\xi_j \leq \mu\xi^2, \qquad \nu, \mu = const > 0,$$

$$\operatorname*{vrai\,max}_{Q_T} \left| \frac{\partial a_{ij}(x, t, u, p)}{\partial p_k} \right| \leq \mu_1, \qquad \left| \frac{\partial a_{ij}}{\partial u}, \frac{\partial a_{ij}}{\partial x_k}, a \right| \leq \varphi(x, t),$$

where $||\varphi||_{2q,2r} \leq \mu_1$ and where $\frac{1}{r} + \frac{n}{2q} < 1$. Then u_x will be Hölder-continuous in (x, t) with a Hölder constant

$$\langle u_x \rangle_{Q'}^{(\alpha, \frac{\alpha}{2})} \leq c\,(\operatorname*{vrai\,max}_{Q_T} |u|, n, \nu, \mu, \mu_1, q, r, d)]$$

where d is the minimum distance from the subdomain Q' on Q_T to the base and laternal surface of the cylinder Q_T, and

$$\alpha = \alpha(\operatorname*{vrai\,max}_{Q_T} |ux|, n, \nu, \mu, \mu_1, q, r).$$

This result as has been mentioned above takes place for the hole class of parabolic equations of second order. The restictions imposed on the functions a_{ij}, a and u being in the nature of the problem t_{00} (in particular, as has been shown in the book on elliptic equations by Uraltzeva and myself, it is impossible to eliminate the requirement that u_x be bounded). This result together with known results on linear equation with smooth coefficients reduce all a priori estimation problems for quasilinear equations of second order to the problem of estimating $\max\limits_{Q_T} |u|$ and $\max\limits_{Q_T} |u_x|$.

I have no time to explain our approach to all these problems. Instead this I'll mention some unsolved problems.

We studied the equations (13, 14) under the following conditions:

1) uniform ellipticity; for the equation (14) this means the restriction

(15) $$\nu(|u|)\,(1 + |p|)^m\xi^2 \leq a_{ij}(x, t, u, p)\,\xi_i\xi_j \leq \mu(|u|)\,(1 + |p|)^m\xi^2;$$

2) continuity (and mostly differentiability) of the function $a_{ij}(x, t, u, p)$, $a_i(x, t, u, p)$ and $a(x, t, u, p)$ in u and p.

These equations have been the object of most works of nonlinear problems up to now. They are well understood now. The methods and results developed in these studies made it possible to investigate certain problems of mechanics in which conditions 1) and 2) do not hold precisely. For example, problems of nonsteady flow through filters, problems connected with Prandtle's equations of boundary-layer flow (for which the form $a_{ij}\xi_i\xi_j$ degenerates, i.e. for which condition 1) not hold), the Stefan problem (in which condition 2) is not satisfied: the functions $a_{ij}(x, t, u, p)$ are discontinuous in u). The other various hydrodynamic problems in which unknown boundaries separate different phases (or flows), may be as the Stefan problem reformulated in terms of equations of the form (13) in which the functions $a_i(x, t, u, p)$ depend discontinuously on u. Attempts to weaken conditions 1) and 2) therefore seems to be of interest. Such attempts might lead to the discovery of new phenomena for parabolic equations and require new methods in addition to the present methods.

Generalisation of results established for equations of the second order to equations of higher order and to systems of equations would also be of interest. We have some results on the system of second order but I shall not formulate them for lack of time.

Show only on the one simple system

$$\vec{v}_t - \Delta\vec{v} + \frac{\partial}{\partial x_k}(v_k\vec{v}) = \vec{f} \qquad (16)$$

$\vec{v} = (v_1(x_1, x_2, x_3, t), v_2, v_3)$ for which the solvability „in the large", is not established. This system may be considered as model for Navie-Stokes system. For system (16) we have no energetic inequality, no maximum principle, in other words, we haven't the first step for the studying of system (16). It is interesting even the question on uniqueness of weak Hopf's solutions for (16) that is the solutions with finite energetic norm $|\vec{v}|Q_T$. I think that the uniqueness not takes place and it seems me that the Hopf's solutions for Navier-Stokes system inself is not unique too. The examples of nonuniqueness mentioned above support my conjecture.