E. A. Coddington

Formally normal ordinary differential operators

In: Valter Šeda (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Bratislava in September 1966. Slovenské pedagogické nakladateľstvo, Bratislava, 1967. Acta Facultatis Rerum Naturalium Universitatis Comenianae. Mathematica, XVII. pp. 65--72.

Persistent URL: http://dml.cz/dmlcz/700198

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

FORMALLY NORMAL ORDINARY DIFFERENTIAL OPERATORS[1)]

E. A. CODDINGTON, Los Angeles

1. Introduction. We shall present some results concerning the spectral theory of ordinary differential operators which commute in a formal way with their formal adjoints. Let L denote the ordinary linear differential expression given by

$$L = \sum_{k=0}^{n} p_k D^k,$$

where D represents the operation id/dx, the p_k are complexvalued functions of class C^∞ on an open interval $a < x < b$ of the real axis, and $p_n(x) \neq 0$ for $a < x < b$. The formal adjoint L^+ of L is given by

$$L^+ = \sum_{k=0}^{n} D^k \bar{p}_k.$$

We say that L commutes formally with L^+, and write $LL^+ = L^+L$, if $LL^+u = L^+Lu$ for all $u \in C^\infty(a, b)$; such an L is said to be formally normal. If L is formally normal we can ask whether it determines, in some natural way, a normal operator in the Hilbert space $L_2(a, b)$, or perhaps in a Hilbert space containing $L_2(a, b)$ as a subspace. We shall indicate below that, in general, this occurs only in rather special cases, and for these cases the spectral theory is easy. We exhibit a large class of formally normal L which determine no normal operators in $L_2(a, b)$, or in any larger Hilbert space. Some details concerning the spectra of these operators are presented. First, we present some abstract results on formally normal operators, which form the basis for the work on ordinary differential operators.

The work reported on here is due to R. E. BALSAM [1], G. BIRIUK and E. A. CODDINGTON [2], and E. A. CODDINGTON [3], [4], [5].

2. Formally normal operators in a Hilbert space. A *formally*

1) This work was supported in part by the National Science Foundation.

normal operator in a Hilbert space $\mathfrak{H}$ is a linear, closed operator with domain $\mathfrak{D}(N)$ dense in $\mathfrak{H}$ such that $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$ and

$$||Nf|| = ||N^*f||, \qquad f \in \mathfrak{D}(N).$$

A *normal operator* in $\mathfrak{H}$ is a formally normal operator N such that $\mathfrak{D}(N) = \mathfrak{D}(N^*)$. For N formally normal define $\overline{N}$ to be the restriction of N^* to $\mathfrak{D}(N)$; thus $\overline{N} = N^*|\mathfrak{D}(N)$. Then $\overline{N} \subset N^*$, in the sense that the graph $\mathfrak{G}(\overline{N})$ of $\overline{N}$ is contained in the graph $\mathfrak{G}(N^*)$ of N^*, and similarly $N \subset \overline{N}^*$. (We note that a symmetric operator in $\mathfrak{H}$ is a formally normal N having the property that $N = \overline{N}$, and a self-adjoint operator is a normal operator such that $N = N^*$.)

If N is formally normal in $\mathfrak{H}$ it can be shown that

$$\mathfrak{D}(\overline{N}^*) = \mathfrak{D}(N) + \mathfrak{M}, \qquad \mathfrak{M} = \nu(I + N^*\overline{N}^*),$$
$$\mathfrak{D}(N^*) = \mathfrak{D}(N) + \overline{\mathfrak{M}}, \qquad \overline{\mathfrak{M}} = \nu(I + \overline{N}^*N^*),$$

where $\nu(A)$ represents the null space of A, and both sums above are direct ones. The following result tells precisely when N has a normal extension in $\mathfrak{H}$. (See [2] and [3]).

Theorem 1. *A formally normal operator N in a Hilbert space $\mathfrak{H}$ has a normal extension N_1 in $\mathfrak{H}$ if and only if*

(1) $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$, *a direct sum,*
(2) $\mathfrak{G}(\overline{N}^*|\mathfrak{M}_1) \perp \mathfrak{G}(\overline{N}^*|\mathfrak{M}_2)$,
(3) $\overline{N}^*\mathfrak{M}_2 = \mathfrak{M}_1$,
(4) $||\overline{N}^*\varphi|| = ||N^*\varphi||, \ \varphi \in \mathfrak{M}_1$,

and

$$\mathfrak{D}(N_1) = \mathfrak{D}(N) + \mathfrak{M}_1, \qquad N \subset N_1 \subset \overline{N}^*.$$

The first two conditions imply that N_1 is a closed operator such that $N \subset N_1 \subset \overline{N}^*$, and the last two guarantee that N_1 is normal. It follows from (3), and the fact that $\overline{N}^*\mathfrak{M} = \overline{\mathfrak{M}}$, that $\mathfrak{M}_1 \subset \mathfrak{M} \cap \overline{\mathfrak{M}}$, and moreover $\dim \mathfrak{M}_1 = \dim \mathfrak{M}_2$. Thus a necessary condition for N to have a normal extension is that $\dim \mathfrak{M}$ be even.

It is not true that every formally normal N in $\mathfrak{H}$ has a normal extension in $\mathfrak{H}$; in fact, this is not even true if N is symmetric. However, every symmetric N has a self-adjoint (and hence normal) extension in the larger Hilbert. space $\mathfrak{H} \oplus \mathfrak{H}$. We ask whether a similar result is valid for formally normal N.

We shall now alter our notation slightly in order to deal with the several Hilbert spaces in what follows. Let us assume N_1 is now a maximal formally normal operator in a Hilbert space $\mathfrak{H}_1$ (N_1 has no proper formally normal extensions in $\mathfrak{H}_1$), and suppose that

$$\mathfrak{D}(\overline{N}_1^*) = \mathfrak{D}(N_1) + \mathfrak{M}^1,$$

where dim $\mathfrak{M}^1$ is finite. It is this case which occurs for ordinary differential operators. In the following we shall refer to two such formally normal operators, N_1 in $\mathfrak{H}_1$, and N_2 in $\mathfrak{H}_2$, with

$$\mathfrak{D}(\overline{N}_2{}^*) = \mathfrak{D}(N_2) + \mathfrak{M}^2,$$

and it will be true that $\mathfrak{M}^1 = \overline{\mathfrak{M}^2}$; and $\mathfrak{M}^2 = \overline{\mathfrak{M}^2}$. We let

$$M_1 = \overline{N}_1{}^*|\mathfrak{M}^1, \qquad M_2 = \overline{N}_2{}^*|\mathfrak{M}^2;$$

thus M_i maps $\mathfrak{M}^i$ into $\mathfrak{M}^i$, $i = 1, 2$. Also, we use the abbreviations

$$\alpha(M_i) = M_i + M_i{}^{*-1},$$
$$\beta(M_i) = M_i{}^*M_i - M_i{}^{*-1}M_i{}^{-1}.$$

In these notations, the following result characterizes when a maximal formally normal operator has a normal extension in a larger Hilbert space (See [2]).

Theorem 2. *A maximal formally normal operator N_1 in a Hilbert space $\mathfrak{H}_1$ has a normal extension $\mathscr{N}_1$ in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ if and only if*

(1) $\mathfrak{M}^1 = \overline{\mathfrak{M}^1}$,

(2) *there exists a formally normal operator N_2 in the Hilbert-space $\mathfrak{H}_2$ such that*

$$\mathfrak{M}^2 = \overline{\mathfrak{M}^2}, \qquad \dim \mathfrak{M}^2 = \dim \mathfrak{M}^1,$$

and a one-to-one map C of $\mathfrak{M}^1$ onto $\mathfrak{M}^2$ satisfying

(3) $\alpha(M_1) + C^*\alpha(M_2)\,C = 0$,

(4) $\beta(M_1) + C^*\beta(M_2)\,C = 0$.

If $\mathscr{N} = N_1 \oplus N_2$, then

$$\mathfrak{D}(\mathscr{N}_1) = \mathfrak{D}(\mathscr{N}) + (I + C)\,\mathfrak{M}^1, \qquad \mathscr{N} \subset \mathscr{N}_1 \subset \mathscr{N}^*.$$

A necessary condition for N_1 to have a normal extension in $\mathfrak{H}$ is given by condition (1), but it is not known whether this is sufficient. An N_2 and C satisfying (2), (3), (4) exist if $M_1M_1{}^* = M_1{}^*M_1$, if $M_1{}^2 = uI$, $|u| = 1$, and in almost all cases if dim $\mathfrak{M}^1 = 2$. Such an N_2 can be defined in terms of a conjugation operator J, which is an operator on $\mathfrak{H}_1$ satisfying $J^2 = I$, and $(Jf, Jg) = (g, f)$ for all $f, g \in \mathfrak{H}_1$. Then $N_2 = J\overline{N}_1J$ on $\mathfrak{H}_2 = \mathfrak{H}_1$ will work in the above cases (See [2]).

3. Formally normal ordinary differential operators. Let us now return to the differential expression

$$L = \sum_{k=0}^{n} p_k D^k, \qquad D = id/dx,$$

which we considered in the Introduction. For our Hilbert space we take $\mathfrak{H}_1 = L_2(a, b)$. We suppose that

$$||Lu|| = ||L^+u||, \qquad u \in C_0^\infty(a, b),$$

where $C_0^\infty(a, b)$ denotes the set of all complex-valued functions of class C^∞ on $a < x < b$ which vanish outside compact subsets of this interval. This restriction on L is equivalent to the condition $LL^+ = L^+L$. Let us define N_1 to be the operator in $\mathfrak{H}_1$ which is the closure of L defined on $C_0^\infty(a, b)$, in the sense that $\mathfrak{G}(N_1)$ is the closure of $\mathfrak{G}(L|C_0^\infty(a, b))$ in $\mathfrak{H}_1 \oplus \mathfrak{H}_1$. This operator N_1 is formally normal in $\mathfrak{H}_1$ and is called the *minimal operator* in $\mathfrak{H}_1$ associated with L. The operator $\bar{N}_1{}^*$ is just L on $\mathfrak{D}(\bar{N}_1{}^*)$, which is the set of all $f \in \mathfrak{H}_1$ such that $f \in C^{n-1}(a, b)$, $f^{(n-1)}$ is absolutely continuous, and $Lf \in \mathfrak{H}_1$. The operator $N_1{}^*$ is described in the same way with L replaced by L^+. Hence $\bar{N}_1$ is L^+ on $\mathfrak{D}(N_1)$. We have

$$\mathfrak{D}(\bar{N}_1{}^*) = \mathfrak{D}(N_1) + \mathfrak{M}^1, \qquad \mathfrak{D}(N_1{}^*) = \mathfrak{D}(N_1) + \overline{\mathfrak{M}}^1,$$

where

$$\mathfrak{M}^1 = \{\varphi \in \mathfrak{H}_1 \,|(LL^+ + I)\, \varphi = 0, L\varphi \in \mathfrak{H}_1\},$$

$$\overline{\mathfrak{M}}^1 = \{\psi \in \mathfrak{H}_1 \,|(LL^+ + I)\, \psi = 0, L^+\psi \in \mathfrak{H}_1\}.$$

Thus we see that $\mathfrak{M}^1$ and $\overline{\mathfrak{M}}^1$ consist of solutions to a homogeneous differential equation of order $2n$, and from this it follows that $0 \leqq \dim \mathfrak{M}^1 = \dim \overline{\mathfrak{M}}^1 \leqq 2n$.

The simplest example occurs when all the coefficients p_k are constants. Here there are three cases:

(i) a, b both finite,
(ii) only one of a, b finite,
(iii) $a = -\infty$, $b = +\infty$.

In all cases $\mathfrak{M}^1 = \overline{\mathfrak{M}}^1$, and $\dim \mathfrak{M}^1$ is $2n$, n, and 0 in cases (i), (ii), (iii) respectively. The N_1 for case (iii) as a normal extension. If n is odd, the N_1 of case (ii) is thus normal, and the N for the other cases have the N of case (iii) as a normal eptension. If n is odd, the N, of case (ii) has no normal eptension in $\mathfrak{H}_1$; see the remarks just after the statement of Theorem 1.

We now interpret Theorem 1 for our differential operator N_1. The conditions of that theorem can be given a more conventional form by means of two bracketed expressions:

$$\langle uv \rangle = (Lu, v) - (u, L^+v), \ u \in \mathfrak{D}(\bar{N}_1{}^*), \ v \in \mathfrak{D}(N_1{}^*),$$

$$[uv] = (Lu, Lv) - (L^+u, L^+v), \ u, v \in \mathfrak{D}(\bar{N}_1{}^*) \cap \mathfrak{D}(N_1{}^*).$$

It can be shown that these expressions depend only on $u, u', \ldots, u^{(n-1)}$ and $v, v', \ldots, v^{(n-1)}$ in the vicinity of a and b, and they are limiting values of certain semi-bilinear forms in these quantities. Theorem 1 can be rephrased in these terms so as to display the domain of a normal extension of N_1 described by certain boundary conditions.

Theorem 3. *The minimal operator N_1 associated with L has a normal extension $\hat{N}_1$ in $\mathfrak{H}_1 = L_2(a, b)$ if and only if* $\dim \mathfrak{M}^1 = 2p$, *and there exist linearly independent $\alpha_1, \ldots, \alpha_p \in \mathfrak{M}^1 \cap \overline{\mathfrak{M}^1}$ satisfying*

$$\langle \alpha_j \alpha_k \rangle = [\alpha_j \alpha_k] = 0, \qquad (j, k = 1, \ldots, p).$$

Then $N_1 \subset \hat{N}_1 \subset \overline{N}_1{}^*$ and

$$\mathfrak{D}(\hat{N}_1) = \{u \in \mathfrak{D}(\overline{N}_1{}^*) \mid \langle u\alpha_j \rangle = 0, j = 1, \ldots, p\}.$$

There is an obvious choice of L whose corresponding N_1 has a good chance of having a normal extension in $\mathfrak{H}_1$, namely those L which can be represented as polynomials in some formally symmetric differential expression Λ:

$$L = \sum_{k=0}^{n} c_k \Lambda^k, \qquad \Lambda = \Lambda^+,$$

where the c_k are complex numbers, some of which may be zero. If the minimal operator for Λ has a self-adjoint extension S in $\mathfrak{H}_1$, then clearly

$$\hat{N}_1 = \sum_{k=0}^{n} c_k S^k$$

will be a normal extension of N_1 in $\mathfrak{H}_1$. In any case, if L is a polynomial in $\Lambda = \Lambda^+$, it will be true that N_1 has a normal extension $\mathcal{N}_1$ in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_1$. This is due to the fact that the minimal operator S_1 for Λ is symmetric and always has a self-adjoint extension S in $\mathfrak{H}$, and then

$$\mathcal{N}_1 = \sum_{k=0}^{n} c\mathcal{S}_k{}^k$$

is a normal extension of N_1 in $\mathfrak{H}$.

We now might ask: can every L be represented as a polynomial in a formally symmetric Λ, and does every formally normal differential operator N_1 have a normal extension in a larger space? For L of order one or two the answer is yes, but for L of order $n \geqq 3$ the answer is no. The simplest example which shows this is the L defined by

$$Lu = u''' + u'' - 3x^{-2}u' + (3x^{-3} - 2x^{-2})\, u,$$

with $\mathfrak{H}_1 = L_2(0, \infty)$. This determines a formally normal N_1 with $\dim \mathfrak{M}^1 = 1$, but with $\dim (\mathfrak{M}^1 \cap \overline{\mathfrak{M}^1}) = 0$. It is maximal formally normal, but not normal, and has no normal extensions in any Hilbert space $\mathfrak{H} \supset \mathfrak{H}_1$. Recall the necessary condition (1) of Theorem 2.

In spite of the fact that not every L, such that $LL^+ = L^+L$, is a polynomial in a formally symmetric Λ, the following interesting result is valid (see [1]).

Theorem 4. *For L of orders 1, 2 or 3 the formally normal operator N_1 has a normal extension $\widehat{N}_1$ in $\mathfrak{H}_1$ if and only if*

$$L = c_3\Lambda^3 + c_2\Lambda^2 + c_1\Lambda + c_0, \qquad \Lambda = \Lambda^+,$$

where the c_k are constants (some of which may be 0), and

$$\widehat{N}_1 = c_3S^3 + c_2S^2 + c_1S + c_0I,$$

for some self-adjoint extension S of the minimal operator for Λ.

We remark that the example mentioned above is typical for a third order L which is not a polynomial in a formally symmetric Λ. Any N_1, for such an L on a maximal interval of definition, is such that it has no normal extension in any Hilbert space $\mathfrak{H} \supset \mathfrak{H}_1$. Examples of higher order operators, to be given in the next section, further support the conjecture that N_1 has a normal extension in some $\mathfrak{H} \supset \mathfrak{H}_1$ if and only if L is a polynomial in some $\Lambda = \Lambda^+$. Also, one might conjecture that an analogue of Theorem 4 is valid for L of arbitrary order.

The spectral theory of normal operators $\mathcal{N}_1$ in $\mathfrak{H} \supset \mathfrak{H}_1$, which have the form

$$\mathcal{N}_1 = \sum_{k=0}^{n} c_k \mathcal{S}^k,$$

where $\mathcal{S}$ is self-adjoint in $\mathfrak{H}$, is completely determined by the spectral theory for $\mathcal{S}$. If $\mathcal{S}$ has the spectral resolution

$$\mathcal{S} = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E(\lambda),$$

then

$$\mathcal{N}_1 = \int_{-\infty}^{\infty} p(\lambda)\, \mathrm{d}E(\lambda), \qquad p(\lambda) = \sum_{k=0}^{n} c_k \lambda^k.$$

Because of Theorem 4, and the remarks following it, we see that results concerning the spectra of self-adjoint extensions of symmetric ordinary differential operators assume added importance.

4. Some special formally normal differential operators. We have investigated in detail a large class of L, for which $LL^+ = L^+L$, but which can not be expressed as polynomials in a formally symmetric Λ. Let m, n be relatively prime positive integers with $m > n \geqq 2$, and let $g = (m-1)(n-1)/2$. We define the differential expressions L_m and L_n by

$$L_m = x^{-m} \prod_{k=0}^{m-1} (\delta - kn + g),$$

$$L_n = x^{-n} \prod_{k=0}^{n-1} (\delta - km + g),$$

where $\delta = xd$, and d stands for $d/\mathrm{d}x$. These operators have the form

$$L_m = d^m + a_1 x^{-1} d^{m-1} + \ldots + a_m x^{-m},$$
$$L_n = d^n + b_1 x^{-1} d^{n-1} + \ldots + b_n x^{-n},$$

where the a_k and b_k are real constants. Let $L = L_m + L_n$ if one of m, n is even; otherwise let $L = iL_m + L_n$. Then it is true that $LL^+ = L^+L$. If N_1 is the minimal operator in $\mathfrak{H}_1 = L_2(0, \infty)$ for L, then N_1 is formally normal, but not normal; moreover it has no normal extension in $\mathfrak{H}_1$ or in any Hilbert space $\mathfrak{H} \supset \mathfrak{H}_1$ (see [5]). The example in Section 4 is the case $m = 3$, $n = 2$.

The specific nature of the spectrum of N_1 has been determined for each pair of integers m, n. Recall that the resolvent set of N_1 is the set $\varrho(N_1)$ of all $\lambda \in \mathbf{C}$ (the complex numbers) such that $(N_1 - \lambda I)^{-1}$ exists as a bounded operator on all of $\mathfrak{H}_1$, and the spectrum $\sigma(N_1) = \mathbf{C} - \varrho(N_1)$. The point spectrum $\sigma_p(N_1)$ is the set of all $\lambda \in \mathbf{C}$ such that $\dim \nu(N_1 - \lambda I) > 0$; the continuous spectrum $\sigma_c(N_1)$ is the set of all $\lambda \in \sigma(N_1)$ such that $N_1 - \lambda I$ is one-to-one, the range of $N_1 - \lambda I$ is dense in $\mathfrak{H}_1$, but is not all of $\mathfrak{H}_1$; the residual spectrum $\sigma_r(N_1)$ is the set of all $\lambda \in \sigma(N_1)$ such that $N_1 - \lambda I$ is one-to-one, and the range of $N_1 - \lambda I$ is not dense. The essential spectrum $\sigma_e(N_1)$ is the set of all $\lambda \in \mathbf{C}$ such that the range of $N_1 - \lambda I$ is not closed.

There are three cases according as

(*a*) m odd, n even,

(*b*) m odd, n odd,

(*c*) m even, n odd.

For cases (*a*) and (*c*) let $p(r) = r^m + r^n$, and for case (*b*) let $p(r) = ir^m + r^n$. The curve C in $\mathbf{C}$, defined by

$$C = \{\lambda \in \mathbf{C} \mid \lambda = p(it), -\infty < t < \infty\},$$

plays an essential role; in fact $\sigma_e(N_1) = C$. In all cases the point spectrum is empty. If $m > 2n$ in cases (*a*) and (*b*), and $m > 3n$ in case (*c*), we have $\sigma(N_1) = \sigma_r(N_1) = \mathbf{C}$. In the remaining situations the spectrum is more interesting, and depends on certain arithmetic relationships between m and n. The set $\mathbf{C} - C$ consists of two components, which we may call I and II, letting I denote that component which contains the positive real axis. In case (*a*), for example, if $n < m < 2n + 1$, $\sigma(N_1) = C \cup I$ if $m = 2k + 1$ with k even, whereas $\sigma(N_1) = C \cup \mathrm{II}$ if k is odd. The distribution of $\sigma(N_1)$ between $\sigma_c(N_1)$ and $\sigma_r(N_1)$ is further complicated. As an example, if $m = 3$, $n = 2$, we have $\sigma(N_1) = C \cup \mathrm{II}$, $\sigma_e(N_1) = \sigma_c(N_1) = C$, $\sigma_r(N_1) = \mathrm{II}$ (see [5]).

REFERENCES

[1] R. E. Balsam, *Normal ordinary differential operators, Dissertation, University of California, Los Angeles (1965).*

[2] G. Biriuk and E. A. Coddington, *Normal extensions of unbounded formally normal operators, J. Math. Mech., 13 (1964), 617–638.*

[3] E. A. Coddington, *Normal extensions of formally normal operators, Pacific J. Math., 10 (1960), 1203–1209.*

[4] E. A. Coddington, *Formally normal operators having no normal extensions, Can. J. Math., 17 (1965), 1030–1040.*

[5] E. A. Coddington, *The spectra of some formally normal ordinary differential operators, to appear.*