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RECENT DEVELOPMENTS IN THE THEORY OF FUNCTION SPACES

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1. Introduction

The word "function spaces" covers nowadays rather different branches and techniques. In our context function spaces means spaces of functions and distributions defined on the real euclidean n-space R_n which are isotropic, non-homogeneous and unweighted. More precisely, this survey deals with the spaces $B^s_{p,q}$ and $F^s_{p,q}$ on R_n which cover Hölder-Zygmund spaces, Sobolev-Slobodeckij spaces, Besov-Lipschitz spaces, Bessel-potential spaces and spaces of Hardy type. First we try to describe how the different approaches are interrelated, inclusively few historical remarks. Secondly, we outline some very recent developments which, by the opinion of the author, not only unify and simplify the theory of function spaces under consideration considerably, but which also may serve a starting point for further studies.

2. How to Measure Smoothness?

Let R_n be the real euclidean n-space. The classical devises to measure smoothness are derivatives and differences. If one wishes to express smoothness not only locally but globally, in our case on R_n, then function spaces, e.g. of L_p-type, seem to be an appropriate tool. We use standard notations for the derivatives D^α and the differences Δ^m_h,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}} \quad \text{if } x = (x_1,\dots,x_n) \in R_n,\ \alpha = (\alpha_1,\dots,\alpha_n), |\alpha| = \sum_{j=1}^n \alpha_j$$

and

$$\Delta^1_h f(x) = f(x+h) - f(x),\ \Delta^m_h = \Delta^{m-1}_h \Delta^1_h$$

if $x \in R^n$, $h \in R_n$, and $m = 2,3,\dots$ Furthermore,

$$\|f|L_p\| = \left(\int_{R_n} |f(x)|^p dx\right)^{1/p},\quad 0 < p \le \infty,$$

with the usual modification if $p = \infty$. Recall that S and S' stand for the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions on R_n and the space of all complex-

-valued tempered distributions on R_n, respectively. Of course, the spaces L_p with $0 < p \le \infty$ have the usual meaning (complex-valued functions).

Definition 1. (i) (Hölder-Zygmund spaces). Let s be a positive number and let m be an integer with $0 < s < m$. Then

$$C^s = \{f \mid f \in L_\infty, \|f|C^s\|_m = \|f|L_\infty\| + \sup_{\substack{x\in R_n \\ 0\neq h\in R_n}} |h|^{-s}|\Delta_h^m f(x)| < \infty\}. \qquad (1)$$

(ii) (Sobolev spaces). Let $1 < p < \infty$ and let m be a natural number. Then

$$W_p^m = \{f \mid f \in L_p, \|f|W_p^m\| = \sum_{|\alpha|\le m} \|D^\alpha f|L_p\| < \infty\}. \qquad (2)$$

Remark 1. Let $0 < s < 1$. Then

$$\|f|C^s\|_1 = \sup_{x\in R_n} |f(x)| + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^2} \qquad (3)$$

are the familiar norms in the Hölder spaces C^s. If s is a positive fractional number, i.e. $0 < s = [s] + \{s\}$ with $[s]$ integer and $0 < \{s\} < 1$ then (3) can be extended by

$$\sum_{0\le|\alpha|\le[s]} \|D^\alpha f|L_\infty\| + \sum_{|\alpha|=[s]} \|D^\alpha f|C^{\{s\}}\|_1 .$$

The corresponding spaces are the well-known Hölder spaces (on R_n) as they had been used since the twenties. It had been discovered by A. Zygmund [29] in 1945 that it is much more effective to use higher differences than derivatives combined with first differences. Definition 1(i) must be understood in this sense. In particular if s is given then all the admissible norms $\|f|C^s\|_m$ are equivalent to each other. The spaces W_p^m have been introduced by S.L. Sobolev [16] in 1936. The derivatives involved must be understood in the sense of distributions.

In the fifties several attemps hade been made to extend the spaces from Definitio 1, to fill the gaps between L_p, W_p^1, W_p^2,... and to replace the sup-norm in (1) by other norms. On the basis of quite different motivations S.M. Nikol'skij introduced in the early fifties the spaces $\Lambda_{p,\infty}^s$ with $s > 0$, $1 < p < \infty$ (we always prefer the notations used below which are different from the original ones) and L.N.Slobodeckij, N.Aronszajn and E.Gagliardo defined the spaces $\Lambda_{p,p}^s$ with $s > 0$, $1 < p < \infty$. The next major step came around 1960. Let F and F^{-1} be the Fourier transform and its inverse on S', respectively. Let

$$I_s f = F^{-1}[(1 + |\xi|^2)^{\frac{s}{2}} Ff], \quad f \in S', -\infty < s < \infty . \qquad (4)$$

Definition 2. (i) (Besov-Lipschitz spaces). Let $s > 0$, $1 < p < \infty$

and $1 \leq q \leq \infty$. Let m be an integer with $m > s$. Then

$$\Lambda^s_{p,q} = \{f \mid f \in L_p, \|f \mid \Lambda^s_{p,q}\|_m = \|f \mid L_p\| + $$
$$+ \left(\int_{R_n} |h|^{-sq} \|\Delta^m_h f(.) \mid L_p\|^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < \infty\}$$

(usual modification if $q = \infty$).

(ii) (Bessel-potential spaces). Let $-\infty < s < \infty$ and $1 < p < \infty$. Then

$$H^s_p = \{f \mid f \in S', \|f \mid H^s_p\| = \|I_s f \mid L_p\| < \infty\} \quad . \tag{6}$$

Remark 2. The Besov-Lipschitz spaces $\Lambda^s_{p,q}$ have been introduced by O.V. Besov [2,3] (following the way paved by S.M.Nikol'skij). They proved to be one of the most successful scales of function spaces. The two sup-norms in (1) (with respect to $x \in R_n$ and $h \in R_n$) are splitted in (5) in an L_p-norm and an L_q-norm. In some sense these spaces are the appropriate extensions of the spaces C^s in the way described above and they fill the gaps between the Sobolev spaces in a reasonable way, although the Sobolev spaces are not special cases of the spaces $\Lambda^s_{p,q}$ if $p \neq 2$. As in the case of the spaces C^s all the admissible norms $\|f \mid \Lambda^s_{p,q}\|_m$ (with different m's) are pairwise equivalent. The spaces H^s_p have been introduced bv A.P.Calderón [5] and N.Aronszajn, K.T. Smith [1]. First we remark that

$$H^s_p = W^s_p \quad \text{if } s = 0,1,2,\ldots \text{ and } 1 < p < \infty .$$

In other words, also the spaces H^s_p fill the gaps between the Sobolev spaces and extend these spaces to negative values of s. But more important, successful method, the Fourier-analytic approach, or the spectral approach, which we discuss in the next section.

3. The Fourier-Analytical Approach

We return to (4) and (6). Let Δ be the Laplacian on R_n and let E be the identity. Recall that

$$(E - \Delta)f = F^{-1}[(1 + |\xi|^2)Ff], \quad f \in S' .$$

More general, the fractoonal powers of $E - \Delta$ are given by

$$(E - \Delta)^{\frac{s}{2}} f = F^{-1}[(1 + |\xi|^2)^{\frac{s}{2}} Ff], \quad f \in S', \quad -\infty < s < \infty .$$

In other words, $f \in H^s_p$ if and only if $(E - \Delta)^{s/2} f \in L_p$. This gives a better feeling what is going on in (6). In particular, smoothness is measured in the Fourier image by the weight-function $g(\xi) = (1 + |\xi|^2)^{s/2}$, and the growth of this weight-function at infinity represents the degree of smoothness. Let $h(\xi)$ be another positive smooth weight-function, not necessarily of the above polynomial type. In order to provide a better understanding of the Fourier-analytical method we dare a bold speculation: If $h_1(\xi)$ and $h_2(\xi)$ are two weight-functions with the same behaviour at infinity then they generate the same smoothness class in the above sense. It comes out that something of this type is correct (via Fourier multiplier theorems), but we shall not try to make this vague assertion more precise. But on the basis of this speculation we try to replace the above weight-function $g(\xi) = (1+|\xi|^2)^{s/2}$ by more handsome weight-functions which offer a greater flexibility. If $|\xi| \sim 2^j$ with $j = 0,1,2,\ldots$ then $g(\xi) \sim 2^{js}$. Hence one can try to replace $g(\xi)$ by a step function $\tilde{g}(\xi)$ with $\tilde{g}(\xi) \sim 2^{js}$ if $|\xi| \sim 2^j$. This replacement is a little bit too crude, but a smooth version of this idea is just what we want. We give a precise formulation. Let $\varphi(\xi) \in S$ with

$$\operatorname{supp} \varphi \subset \{\xi \,|\, \tfrac{1}{2} \le |\xi| \le 2\}$$

and

$$\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) = 1 \quad \text{if} \quad \xi \neq 0 .$$

Functions with these properties exist. Let $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ if $j = 1,2,3,\ldots$ and $\varphi_0(\xi) = 1 - \sum_{j=1}^{\infty} \varphi_j(\xi)$. Then $\varphi_0(\xi)$ has also a compact support. The desired substitute of $(1+|\xi|^2)^{\frac{s}{2}}$ is now given by $\sum_{j=0}^{\infty} 2^{js}\varphi_j(\xi)$. We introduce the pseudodifferential operators

$$\varphi_j(D)f(x) = F^{-1}[\varphi_j(\xi)Ff](x), \; x \in R_n, \; j = 0,1,2,\ldots, f \in S'. \quad (7)$$

This makes sense because by the Paley-Wiener-Schwartz theorem $\varphi_j(D)f(x)$ is an analytic function in R_n for any $f \in S'$. Furthermore, by a theorem of Paley-Littlewood type we have

$$\|f\,|H_p^s\| \sim \|(\sum_{j=0}^{\infty} |2^{js}\varphi_j(D)f(.)|^2)^{1/2}|L_p\|, \; -\infty < s < \infty, \; 1 < p < \infty, \quad (8)$$

(in the sense of equivalent norms). This is the substitute we are looking for. Now we can ask questions.Does it make sense to replace the l_2-norm in (8) by an l_q-norm (or quasi-norm), $0 < q \le \infty$? Is it reasonable to interchange the roles of L_p and l_2 (or more general l_q) in (8)?

Definition 3. (i) Let $-\infty < s < \infty$, $0 < p \le \infty$ and $0 < q \le \infty$. Then

$$B_{p,q}^s = \{f \,|\, f \in S', \|f\,|B_{p,q}^s\|_\varphi = (\sum_{j=0}^{\infty} 2^{jsq}\|\varphi_j(D)f(.)|L_p\|^q)^{1/q} < \infty\}, \quad (9)$$

(usual modification if $q = \infty$).

(ii) Let $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \le \infty$. Then

$$F_{p,q}^s = \{f \,|\, f \in S', \|f\,|F_{p,q}^s\|_\varphi = \|(\sum_{j=0}^{\infty} 2^{jsq}|\varphi_j(D)f(.)|^q)^{1/q}|L_p\| < \infty\} \quad (10)$$

(usual modification if $q = \infty$).

Remark 3. For all admissible values s,p,q the spaces $B^s_{p,q}$ and $F^s_{p,q}$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$), and they are independent of the chosen function φ (in the sense of equivalent quasi-norms). Maybe this fact is not so astonishing if p and q are restricted by $1 < p < \infty$ and $1 < q < \infty$, because in those cases the Fourier multiplier theory for L_p with $1 < p < \infty$ and its vector-valued counterparts can be taken as hints that something of this type may be valid. But it was a big surprise, also for the creators of this theory, that these definitions make sense even if $0 < p \leq 1$ (and $0 < q \leq 1$). The only exception is $p = \infty$ in the case of the spaces $F^s_{p,q}$ (but even in this case one can do something after appropriate modifications). The above definition of the spaces $B^s_{p,q}$ is due to J.Peetre [11,12]. The spaces $F^s_{p,q}$ have been introduced by the author [19], P.I.Lizorkin [10] and J.Peetre [13]. Fro the greater part of the theory of these spaces a restriction to $p \geq 1$, $q \geq 1$ would be artifical. But from a technical point of view such a restriction often simplifies the proofs because one has the elaborated technique of Banach space theory at hand (and one avoids a lot of pitfalls which are so abundant if $p<1$). Systematic treatments of the theory of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ have been given in [14] (mostly restricted to $B^s_{p,q}$ with $1 \leq p \leq \infty$) and [23] (with [21,22] as forerunners, cf. also [20]). Again one can ask questions. What is the use of these spaces? What is the connection of these spaces and those ones introduced in Section 2? As far as the latter question is concerned one has the following answer.

Theorem 1. (i) Let $s > 0$. Then

$$C^s = B^s_{\infty,\infty} . \tag{11}$$

(ii) Let $1 < p < \infty$ and $-\infty < s < \infty$. Then

$$H^s_p = F^s_{p,q} , \tag{12}$$

(in particular, $W^m_p = F^s_{p,2}$ if $m = 0,1,2,...$ and $1 < p < \infty$).

(iii) Let $s > 0$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Then

$$\Lambda^s_{p,q} = B^s_{p,q} . \tag{13}$$

(iv) Let $0 < p < \infty$. Then $F^0_{p,2}$ is a (non-homogeneous) space of Hardy type.

Remark 4. Proofs may be found in [23], cf. also Sections 6 and 7.

4. Points Left Open

The Fourier-analytical approach proved to be very useful in connection with applications to linear and non-linear partial differential equations, cf. [20,23] as far as linear equations are concerned. In the recently developed method of para-multiplications by J.M.Bony and Y.Meyer (in order to obtain local and microlocal smoothness assertions for non-linear partial differential equations) characterizations of type (11) play a crucial role. An extension of these methods to the full scales $B^s_{p,q}$ and $F^s_{p,q}$ has been given by T.Runst [15] (there one can also find the necessary references to the papers by Bony, Meyer).

There is no claim that this paper gives a systematic description of the history of those function spaces which are treated here. We omitted few important developments. But we wish to mention at least few key-words and some milestone-papers. Interpolation theory plays a crucial role in the theory of function spaces since the sixties. The outstanding papers are those ones of J.-L.Lions, J.Peetre [9] and A.P. Calderón [6]. A systematic approach to the theory of function spaces from the standpoint of interpolation theory has been given in [20]. Another important approach to the theory of function spaces is the real variable method in the theory of Hardy spaces and the elaboration of the technique of maximal functions. The milestone-paper in this field is C.Fefferman, E.M.Stein [7].

5. Harmonic and Thermic Extensions

The interest in Hardy spaces has its origin in complex function theory: traces of holomorphic functions in the unit disc or the upper half-plane on the respective boundaries. A generalization of this idea yields a characterization of functions and distributions of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ on R_n as traces of harmonic functions or temperaturs in $R^+_{n+1} = \{(x,t) \mid x \in R_n,\ t > 0\}$ on the hyperplane $t = 0$, which is identified with R_n. We reformulate this problem as follows. Let $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ be the Laplacian in R_n and let $f \in B^s_{p,q}$ or $f \in F^s_{p,q}$. What can be said (in the sense of characterizing properties) about the solutions $u(x,t)$ and $v(x,t)$ of the problems

$$\left(\frac{\partial^2 u}{\partial t^2} + \Delta u\right)(x,t) = 0 \text{ if } (x,t) \in R^+_{n+1};\ u(x,0) = f(x) \text{ if } x \in R_n \quad (14)$$

(harmonic extension) and

$$\left(\frac{\partial v}{\partial t} - \Delta v\right)(x,t) = 0 \text{ if } (x,t) \in R^+_{n+1};\ v(x,0) = f(x) \text{ if } x \in R_n \quad (15)$$

(thermic extension)? At least in a formal way the solutions u(x,t) and

$v(x,t)$ are known,

$$u(x,t) = P(t)f(x) = c \int_{R_n} \frac{t}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} f(y)dy, \quad x \in R_n, \ t>0 \qquad (16)$$

(Cauchy-Poisson semigroup) and

$$v(x,t) = W(t)f(x) = ct^{-\frac{n}{2}} \int_{R_n} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \quad x \in R_n, \ t > 0 \qquad (17)$$

(Gauss-Weierstrass semigroup). If $f \in S'$ is given, then (17) makes sense. Furthermore, (16) must be understood in the following theorem via limiting procedures. If a is a real number we put $a_+ = \max(0,a)$.

Theorem 2. Let $\varphi_0 \in S$ with $\varphi_0(0) \neq 0$.

(i) Let $-\infty < s < \infty$, $0 < p \le \infty$, and $0 < q \le \infty$. Let k and m be non-negative integers with $k > n(\frac{1}{p} - 1)_+ + \max(s, n(\frac{1}{p} - 1)_+)$ and $2m > s$. Then

$$\|\varphi_0(D)f|L_p\| + \left(\int_0^1 t^{(k-s)q} \left\|\frac{\partial^k P(t)f}{\partial t^k}|L_p\right\|^q \frac{dt}{t}\right)^{\frac{1}{q}} \qquad (18)$$

and

$$\|\varphi_0(D)f|L_p\| + \left(\int_0^1 t^{(m-\frac{s}{2})q} \left\|\frac{\partial^m W(t)f}{\partial t^m}|L_p\right\|^q \frac{dt}{t}\right)^{\frac{1}{q}} \qquad (19)$$

(modification if $q = \infty$) are equivalent quasi-norms in $B^s_{p,q}$. If $s > n(\frac{1}{p} - 1)_+$ then $\|\varphi_0(D)f|L_p\|$ in (18),(19) can be replaced by $\|f|L_p\|$.

(ii) Let $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \le \infty$. Let k and m be non-negative integers with $k > \frac{n}{\min(p,q)} + \max(s, n(\frac{1}{p} - 1)_+)$ and $2m > s$. Then

$$\|\varphi_0(D)f|L_p\| + \left\|\left(\int_0^1 t^{(k-s)q} \left|\frac{\partial^k P(t)f}{\partial t^k}(\cdot)\right|^q \frac{dt}{t}\right)^{\frac{1}{q}}|L_p\right\| \qquad (20)$$

and

$$\|\varphi_0(D)f|L_p\| + \left\|\left(\int_0^1 t^{(m-\frac{s}{2})q} \left|\frac{\partial^m W(t)f}{\partial t^m}(\cdot)\right|^q \frac{dt}{t}\right)^{\frac{1}{q}}|L_p\right\| \qquad (21)$$

(modification if $q = \infty$) are equivalent quasi-norms in $F^s_{p,q}$. If $s > n(\frac{1}{p} - 1)_+$ then $\|\varphi_0(D)f|L_p\|$ in (20), (21) can be replaced by $\|f|L_p\|$.

Remark 5. Characterizations of the above type have a long history. As far as the classical Besov-Lipschitz spaces $\Lambda^s_{p,q}$ and the Bessel-potential spaces H^s_p are concerned the first comprehensive treatment in the sense of the above theorem has been given by M.H.Taibleson [18], cf. also T.M.Flett [8]. In this context we mention also the books by P.L.Butzler, H.Berens [4] and E.M. Stein [17] where one can find many informations about characterizations of the above type (for the classical space) and the semigroups from (16) and (17), cf. also [20, 2.5.2, 2.5.3]. More recent results (characterizations of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ in the sense of the above theorem) have been obtained by G.A.Kaljabin, B.-H.Qui and the author. The above formulation has been taken over from [25] (cf. also [23, 2.12.2] where we also gave references to the papers by B.-H.Qui and G.A.Kaljabin).

6. Unified Approach

Up to this moment we said nothing how to understand that the apparently rather different approaches via derivatives, differences, Fourier-analytical decompositions, harmonic and thermic extensions, always yield the same spaces $B^s_{p,q}$ and $F^s_{p,q}$. In [23] we proved equivalence assertions of the above type mostly by rather specific arguments, cf. also [14,22]. But recently it became clear that there exists a unified approach which covers all these methods, at least in principle, and which sheds some light on the just-mentioned problem. We follow [25] where [24] may be considered as a first step in this direction. The basic idea is to extend the admissible functions φ and φ_j in (7) and (9), (10), such that corresponding (quasi-)norms in the sense of (9), (10) cover automatically characterizations of type (18), (19) and (5). We recall that

$$\varphi(tD)f(x) = F^{-1}[\varphi(t.)Ff](x) = ct^k \frac{\partial^k P(t)}{\partial t^k} f(x) \text{ if } \varphi(\xi) = |\xi|^k e^{-|\xi|} \tag{22}$$

and

$$\varphi(\sqrt{t}\, D)f(x) = ct^m \frac{\partial^m W(t)f(x)}{\partial t^m} \text{ if } \varphi(\xi) = |\xi|^{2m} e^{-|\xi|^2} \tag{23}$$

Furthermore we remark that the discrete quasi-norms in (9) and (10) have always continuous counterparts, i.e.

$$\|\varphi_0(D)f|L_p\| + \left(\int_0^1 t^{-sq}\|\varphi(td)f(.)|L_p\|^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{24}$$

is the continuous substitute of the quasi-norm in (9) and

$$\|\varphi_0(D)f|L_p\| + \left\|\left(\int_0^1 t^{-sq}|\varphi(tD)f(.)|^q \frac{dt}{t}\right)^{\frac{1}{q}}|L_p\right\| \tag{25}$$

is the continuous substitute of the quasi-norm in (10). This replacement of "discrete" quasi-norms by "continuous" ones is a technical matter and has nothing to do with the extension of the class of admissible φ's which we have in mind. If one puts (22),(23) in (24),(25) then one obtains (18)-(21). Of course one has to clarify under what conditions for the parameters involved this procedure is correct. However before giving some details we ask how to incorporate derivatives and differences in this Fourier-analytical concept. We have

$$\varphi(D)f(x) = cD^{\alpha}\Delta_h^m f(x) \text{ if } \phi(\xi) = \xi^{\alpha}(e^{i\xi h}-1)^m, \tag{26}$$

with $\alpha = (\alpha_1,\dots,\alpha_n)$, m natural number, and $\xi h = \sum_{j=1}^{n} \xi_j h_j$, $\xi^{\alpha} = \xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$. The three functions φ in (29),(23),(26) have in common that they tend to tero if $|\xi| \to 0$ (even if $\alpha = 0$ in (26)). In addition the functions φ from (22),(23) have the same property if $|\xi| \to \infty$. If one compares these functions φ with the function φ from Section 3 used in Definition 3 then it seems to be at least plausible that one can substitute φ in (9),(10) by the functions φ from (22),(23) if k and m are chosen sufficiently large. As for the function φ form (26) this question is more delicate. First one has no decay if ξ tends to infinity and secondly one has not only to handle an isolated function φ but a family of functions parametrized by $h \in R_n$ (and, maybe, by α). We return to these questions later on and formulate a result which covers in principle all cases of interest.

Let $h(x) \in S$ and $H(x) \in S$ with supp $h \subset \{y\,|\,|y| \le 2\}$, supp $H \subset \{y\,|\,\frac{1}{4} \le |y| \le 4\}$, $h(x) = 1$ if $|x| \le 1$, and $H(x) = 1$ if $\frac{1}{2} \le |x| \le 2$.

Theorem 3. Let $0 < p \le \infty$, $0 < q \le \infty$ and $-\infty < s < \infty$. Let s_0 and s_1 be two real numbers with

$$s_0 + n(\tfrac{1}{p} - 1)_+ < s < s_1 \quad\text{and}\quad s_1 > n(\tfrac{1}{p} - 1)_+ \,. \tag{27}$$

Let $\varphi_0(\xi)$ and $\varphi(\xi)$ be two infinitely differentiable complex-valued functions on R_n and $R_n - \{0\}$, respectively, which satisfy the Tauberian conditions

$$|\varphi_0(\xi)| > 0 \text{ if } |\xi| \le 2 \text{ and } |\varphi(\xi)| > 0 \text{ if } \tfrac{1}{2} \le |\xi| \le 2. \tag{28}$$

let $\tilde{p} = \min(1,p)$ and

$$\int_{R_n} |(F^{-1}\frac{\varphi(z)h(z)}{|z|^{s_1}})(y)|^{\tilde{p}}dy < \infty, \tag{29}$$

$$\sup_{m=1,2,..} 2^{-ms_0\tilde{p}} \int_{R_n} |(F^{-1}\varphi(2^m.)H(.))(y)|^{\tilde{p}}dy < \infty, \tag{30}$$

and (30) with φ_0 instead of φ. Then

$$\|\varphi_0(D)f\,|L_p\| + (\int_0^1 t^{-sq}\|\varphi(tD)f(.)\,|L_p\|^q \frac{dt}{t})^{\frac{1}{q}} \tag{31}$$

(modification if $q = \infty$) is an equivalent quasi-norm in $B^s_{p,q}$.

Remark 6. This formulation coincides essentially with Theorem 3 in [25]. Of course, $\varphi(tD)f = F^{-1}[\varphi(t.)Ff](x)$ and (31) coincides with (24). This theorem has a direct counterpart for the spaces $F^s_{p,q}$. Furthermore there are some modifications (both for $B^s_{p,q}$ and $F^s_{p,q}$) where not only a single function φ but families of these functions are involved, cf. the considerations in front of the above theorem. Maybe the crucial conditions (29) and (30) look somewhat complicated and seem to be hard to check. But this is not the case, in particular for functions of type (26) the formulations (29),(30) are well adapted. Furthermore, if one uses

$$\|F^{-1}\lambda|L_v\| \le c\|\lambda|H_2^\delta\|, \quad 0 < v \le 1, \quad \delta > n(\frac{1}{v} - \frac{1}{2}), \qquad (32)$$

then one can replace (29),(30) by more handsome-looking conditions, where only Bessel-potential spaces H_2^δ (or even Sobolev spaces W_2^δ) are involved.

Remark 7. Theorem 2 follows from Theorem 3 and its $F^s_{p,q}$-counterpart. One has to use the functions φ from (22),(23).

7. Characterizations via Differences

In principle one can put φ from (26) in Theorem 3 and its $F^s_{p,q}$-counterpart. One can calculate under what conditions for the pamaterers (29),(30) are satisfied. However as we pointed out in front of Theorem 3 one has to modify Theorem 3, because one needs now theorems with families of functions φ instead of a single function φ. This can be done, details may be found in [25]. We formulate a result what can be obtained on this way.

Theorem 4. (i) Let $0 < p \le \infty$, $0 < q \le \infty$ and $n(\frac{1}{p} - 1)_+ < s < m$, where m is a natural number. Then

$$\|f|L_p\| + \Big(\int_{|h|\le 1} |h|^{-sq}\|\Delta_h^m f|L_p\|^q \frac{dh}{|h|^n}\Big)^{\frac{1}{q}} \qquad (33)$$

(modification if $q = \infty$) is an equivalent quasi-norm in $B^s_{p,q}$.

(ii) Let $0 < p < \infty$, $0 < q \le \infty$ and $\frac{n}{\min(p,q)} < s < m$, where m is a natural number. Then

$$\|f|L_p\| + \Big\|\Big(\int_{|h|\le 1} |h|^{-sq}|(\Delta_h^m f)(.)|^q \frac{dh}{|h|^n}\Big)^{\frac{1}{q}}|L_p\Big\| \qquad (34)$$

(modification if $q = \infty$) is an equivalent quasi-norm in $F^s_{p,q}$.

Remark 8. We refer for details to [25] where we proved many other theorems of this type via Fourier-analytical approach from Section 6 and few additional considerations. However the theorem itself is not new, it may be found in [23, 2.5.10, 2.5.12]. But the proof in [23] is more complicated and not so clearly based on Fourier-analytical results in the sense of Theorem 3. On the basis of Theorem 4 one has now also a better understanding of (11) and (13). We prefered in the above theorem a formulation via differences only. But one can replace some differences by derivatives, as it is also suggested by (26).

8. The Local Approcah

The original Fourier-analytical approach as described in Section 3 does not reflect the local nature of the spaces $B^s_{p,q}$ and $F^s_{p,q}$. If $x \in R_n$ is given then one needs a knowledge of f on the whole R_n in order to calculate $\varphi_j(D)f(x)$ in (7). This stands in sharp contrast to the derivatives $D^\alpha f(x)$ and the differences $\Delta^m_h f(x)$ with $|h| \leq 1$ as they have been used above. However the extended Fourier-analytical method as described in Section 6 gives the possibility to combine the advantages of the original Fourier-analytical approach and of a strictly local procedure. We give a description. Let $k_0 \in S$, and $k \in S$ with

$$\operatorname{supp} k_0 \subset \{y \mid |y| \leq 1\}, \quad \operatorname{supp} k \subset \{y \mid |y| \leq 1\},$$
$$(Fk_0)(0) \neq 0 \quad \text{and} \quad (Fk)(0) \neq 0.$$

Let $k_N = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\right)^N k$, where N is a natural number. We introduce the means

$$K(k_N,t)f(x) = \int_{R_n} k_N(y)f(x+ty)dy, \quad x \in R_n, \ t > 0, \tag{35}$$

where now $N = 0,1,2,\ldots$ This makes sense for any $f \in S'$.

Theorem 5. (i) Let $-\infty < s < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $0 < \varepsilon < \infty$, $0 < r < \infty$ and $2N > \max\left(s, n\left(\frac{1}{q} - 1\right)_+\right)$. Then

$$\|K(k_0,\varepsilon)f \mid L_p\| + \left(\int_0^r t^{-sq} \|K(k_N,t)f \mid L_p\|^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{36}$$

(modification if $q = \infty$) is an equivalent quasi-norm in $B^s_{p,q}$.

(ii) Let $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leq \infty$. Let $0 < \varepsilon < \infty$, $0 < r < \infty$ and $2N > \max\left(s, n\left(\frac{1}{p} - 1\right)_+\right)$. Then

$$\|K(k_0,\varepsilon)f \mid L_p\| + \left\|\left(\int_0^r t^{-sq} |K(k_N,t)f(.)|^q \frac{dt}{t}\right)^{\frac{1}{q}} \mid L_p\right\| \tag{37}$$

(modification if $q = \infty$) is an equivalent quasi-norm in $F^s_{p,q}$.

Remark 9. It comes out that the above theorem can be obtained from Theorem 3 and its $F^s_{p,q}$-counterpart. On the other hand it is clear that (35) describes a local procedure.

Remark 10. With the help of Theorem 5 one can simplify and unify several proofs in [23], cf. e.g. [26]. But it is also an appropriate tool to handle psudodifferential operators, cf. [28], and to introduce spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type on complete Riemannian manifolds (which are not necessarily compact), cf. [27],

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